Math 273, Final Exam Solutions

1. Find the solution of the differential equation $y' = y + e^{-x}$ that satisfies the condition $y(x) \to 0$ as $x \to +\infty$.

   SOLUTION: $y = y_H + y_P$ where $y_H = ce^x$ is a solution of the homogeneous equation and y_P is a particular integral, which must have the form Ae^{-x}. Substituting this into the equation we find $A = -1/2$ and the required result is
   $$y(x) = -(1/2)e^{-x}.$$

2. Consider the differential equation
   $$Lu \equiv (pu')' + qu = 0 \tag{1}$$
   on the symmetric interval $[-1, 1]$. Suppose p is C^1 and does not vanish on that interval, that q is continuous there and they are even functions: $p(-x) = p(x), \quad q(-x) = q(x)$.

   (a) Show that there exists a basis of solutions u, v of the equation $Lu = 0$ such that u is even and v is odd.

   SOLUTION: If $u(x)$ is any solution, put $\tilde{u}(x) = u(-x)$. Then $\tilde{u}'(x) = -u'(-x)$ and $\tilde{u}''(x) = u''(-x)$, so
   $$L\tilde{u}(x) = p(x)u''(-x) - p'(x)u'(-x) + q(x)u(-x) =$$
   $$p(-x)u''(-x) + p'(-x)u'(-x) + q(-x)u(-x) = (Lu)(-x) = 0,$$
   in view of the evenness of p and q. In other words, $\tilde{u}$ is a solution of the equation if u is. Consider now the standard basis consisting of solutions u and v such that
   $$u(0) = 1,\ u'(0) = 0; \quad v(0) = 0,\ v'(0) = 1.$$
   The first of these is even since if $\tilde{u}(x) = u(-x)$, it satisfies the same initial conditions as u as well as the differential equation and by uniqueness $u(-x) = u(x)$. The second is odd since if $\tilde{v}(x) = -v(-x)$, it satisfies the same initial conditions as v as well as the differential equation, and therefore $v(-x) = -v(x)$.

   (b) Show that, for the *boundary-value* problem consisting of equation (1) together with the conditions
   $$u(-1) = u(1),\ u'(-1) = -u'(1),$$

there exists a nontrivial solution.

SOLUTION: The even solution of the first part satisfies these conditions.

3. Let $A(t)$ be a continuous, $n \times n$ matrix function defined for $t \in R$, and consider the linear system

$$\dot{x} = A(t)x. \qquad (2)$$

(a) Explain what is meant by a fundamental matrix solution $\Phi(t)$ of equation (2).
SOLUTION: It is a matrix whose columns form a linearly independent set of vector solutions of equation (2). Alternatively, it is a matrix-valued solution of $\dot{\Phi} = A(t)\Phi$ which is nonsingular.

(b) Derive a matrix differential equation satisfied by the transposed matrix $\Phi^T(t)$.
SOLUTION: Since $\dot{\Phi} = A(t)\Phi$, taking transposes gives $\dot{\Phi}^T = \Phi^T A(t)^T$.

(c) Derive a matrix differential equation satisfied by the inverse matrix $\Phi^{-1}(t)$.
SOLUTION: $\Phi\Phi^{-1} = I$, a constant, so differentiating gives

$$0 = \frac{d\Phi}{dt}\Phi^{-1} + \Phi\frac{d\Phi^{-1}}{dt} = A(t) + \Phi\frac{d\Phi^{-1}}{dt},$$

so $d\Phi^{-1}/dt = -\Phi^{-1}A(t)$.

4. Consider the analytic initial-value problem

$$w'' = 2w^3, \quad w(0) = 1, \; w'(0) = 1.$$

(a) Assuming there is a power-series solution valid near $z = 0$, find it through terms of order z^4.
SOLUTION: Differentiating either side of the equation twice gives

$$w''' = 6w^2w' \text{ and } w^{iv} = 6w^2w'' + 12ww'^2.$$

Using these and the initial data give, at $z = 0$,

$$w'' = 2, w''' = 6, w^{iv} = 24.$$

The Macalurin expansion then shows that $w(z) = 1 + z + z^2 + z^3 + z^4 + \ldots$.

(b) Find an explicit, exact solution and thereby determine the radius of convergence of the power series of part 4a.
SOLUTION: The exact solution is $w = 1/(1-z)$ which can be verified. It can also be derived:

$w'w'' = 2w^3w'$ implies $w'^2 = w^4$ in view of the initial conditions;

this implies $w' = w^2$ which leads to the same result.

5. For the equation $z^2w'' + \sin(3z)w' + \cos(3z)w = 0$

   (a) Show that the origin is a regular singular point.
   SOLUTION: This has the structure of such an equation if $\sin(3z) = zP(z)$ where P is analytic at the origin. It is with

$$P(z) = 3\left(1 - (3z)^2/3! + (3z)^4/5! - \cdots\right).$$

   (b) Find the possible indices μ for the Frobenius-type expansions $z = 0$ (you do *not* need to work out the recursion relation).
   SOLUTION: With $P_0 = 3$ and $Q_0 = 1$ the indicial equation $\mu^2 + (P_0 - 1)\mu + Q_0 = 0$ has the solution $\mu = -1$ with multiplicity two.

   (c) Does a second solution involve a logarithmic multiple of the first? Explain.
   SOLUTION: Yes, this is always the case when the indices are the same.

6. Consider the system $\dot{x} = f(t, x)$ where f is C^1 on a domain $D \in R^{(n+1)}$. Let $\phi(t, t_0, y)$ be the solution of this equation taking on the value y at the time t_0, on an interval $a \le t \le b$.

   (a) The vector function $u(t) = \partial\phi(t, t_0, y)/\partial y_k$ exists on this interval and satisfies a linear differential equation. Find the initial-value problem for u. You do not need to justify your steps.
   SOLUTION: The equation is $\dot{u} = A(t)u$ where $A(t) = (Df)(t, \phi(t, t_0, y))$, where Df is the Jacobian matrix of f. Since

$$\partial\phi(t_0, t_0, y)/\partial y_k = \partial y/\partial y_k = e_k,$$

   where e_k is the kth, standard unit vector, the initial condition is $u(t_0) = e_k$.

(b) The vector function $v(t) = \partial\phi(t, t_0, y)/\partial t_0$ exists on this interval and satisfies a linear differential equation. Find the initial-value problem for v. You do not need to justify your steps.
SOLUTION: The differential equation for v is the same as that for u. To find the initial data, note that

$$\phi(t, t_0, y) = y + \int_{t_0}^{t} f(s, \phi(s, t_0, y)\, ds,$$

and differentiating with respect to t_0 followed by taking the limit as $t \to t_0$ gives $v(t_0) = -f(t_0, y)$.

7. Consider the system $\dot{x} = -x + 2xy, \dot{y} = -y + x^2$.

(a) Find the equilibrium solutions.
SOLUTION: $(x, y) = (0, 0), (1/\sqrt{2}, 1/2), (-1/\sqrt{2}, 1/2)$.

(b) Determine their stability.
SOLUTION: The Jacobian matrix is

$$\left(\begin{array}{cc} 2y - 1 & 2x \\ 2x & -1) \end{array} \right).$$

At $(x, y) = (0, 0)$ the eigenvalues are each -1 so this point is asymptotically stable. At the points $(x, y) = (\pm 1/\sqrt{2}, 1/2)$ they are 1 and -2 so these points are unstable.

(c) Can this system have a nonconstant, periodic orbit? Explain.
SOLUTION: It cannot, because this is a gradient system $(\dot{x}, \dot{y}) = \nabla\phi$ where $\phi = -(x^2 + y^2)/2 + x^2 y$.

8. Consider the initial-value problem $\dot{x} = f(t, x), \quad x(0) = y$, for a *scalar*, i.e., one-dimensional, equation for which the function f is defined for all $t \geq 0$ and all real x. Assume that a unique solution $x = \phi(t, y)$ exists for any y and all $t > 0$. Suppose further that

(a) $f(t, 0) = 0$ for all $t \geq 0$, and

(b) $\phi(t, y) \to 0$ as $t \to \infty$, for any y.

Define stability of the origin as usual: it is stable if, given any $\epsilon > 0$ there is $\delta > 0$ such that $|\phi(t, y)| < \epsilon$ for all $t > 0$ provided $|y| < \delta$.

Show that the origin is stable.
SOLUTION: Choose positive and negative values of y, say $y_0 = \pm 1$.

Consider first positive initial data, $y > 0$. The solution $x = \phi(t, y)$ then remains positive, since the axis $x = 0$ is invariant and cannot be reached if $y > 0$. Likewise, if $0 < y < 1$, the graph of the solution $\phi(t, y)$ must lie below the graph of the solution $\phi(t, 1)$. The latter function tends to zero so, given $\epsilon > 0$ there is $T > 0$ such that $0 < \phi(t, 1) < \epsilon$ if $t \geq T$. Since $\phi(t, 0) = 0$, by continuity with respect to initial data, we can choose δ in the interval $(0, 1)$ so that $0 < y < \delta$ implies that $\phi(t, y) < \epsilon$ on $[0, T]$. Its graph lies below that of $\phi(t, 1)$ on $[T, \infty)$, completing the proof for $y > 0$, and the same argument applies for $y < 0$.