Answers to Selected Problems in Chapter 1

1. Problem Set 1.2.1

   - 1) The standard form is
   $$\dot{x} = \frac{1-x^2}{1+t^2}.$$
   - 5) The antiderivative is $x(t) = (1/3)t^3 - e^t + C$ so, applying the initial condition, one finds
   $$x(t) = (1/3)t^3 - e^t + 3.$$
   - 7) Applying formula (1.23) one finds $x(t) = K\exp(-\alpha t) + \gamma/\alpha$ so, evaluating the constant K with the aid of the initial data, one finds
   $$x(t) = (1-\gamma/\alpha)e^{-\alpha t} + \gamma/\alpha.$$
   - 9)
   $$\frac{d}{dt}(c_1x_1 + c_2x_2) = c_1\frac{dx_1}{dt} + c_2\frac{dx_2}{dt} = c_1k(t)x_1 + c_2k(t)x_2$$
   $$= k(t)(c_1x_1 + c_2x_2).$$
   - 10) If $\dot{x} \equiv 0$ then $x = \gamma/\alpha$. The solution for Problem 7 for arbitrary initial data $x(0) = k$ is
   $$x(t) = (k-\gamma/\alpha)e^{-\alpha t} + \gamma/\alpha,$$
   which tends to γ/α as $t \to \infty$ since $\alpha > 0$.
   - 11) $\frac{\partial}{\partial s}f(s-t) = f'(s-t)$ and $\frac{\partial}{\partial t}f(s-t) = -f'(s-t)$, so the result follows.
   - 14) The solution is $x(t) = x_0\exp(\sin t)$; it is periodic.

2. Problem Set 1.3.1

   - 1) If $\dot{x} = k_0x^{1+\epsilon}$ then
   $$x^{-(1+\epsilon)}\frac{dx}{dt} = \frac{d}{dt}\left(-x^{-\epsilon}/\epsilon\right) = k_0,$$

so $x^{-\epsilon} = C - k_0 t$ where C is a constant. Evaluating this gives

$$x(t) = \left(x_0^{-\epsilon} - \epsilon k_0 t\right)^{-(1/\epsilon)},$$

which tends to infinity as t tends to $t_* = x_0^{-\epsilon}/\epsilon k_0$.

- 4) The formula gives $y(x) = C \exp(\ln x) = Cx$ where C is a constant. The substitution $y = xu(x)$ give $xu' = 0$, so $u = C$.
- 8) Rewrite the equation as

$$\frac{1}{u^2 - c^2} \frac{du}{dt} = 1.$$

Since
$$\frac{1}{u^2 - c^2} = -\frac{1}{2c}\left(\frac{1}{c+u} + \frac{1}{c-u}\right),$$
this can be integrated. Assume for definiteness that u_0 lies in the interval $(-c, c)$. Then the integration gives

$$\frac{c+u}{c-u} = ke^{-2ct}, \quad k = \frac{c+u_0}{c-u_0},$$

or
$$u(t) = c\frac{ke^{-2ct} - 1}{ke^{-2ct} + 1}.$$
In this case, $u \to -c$ as $t \to +\infty$ and $u \to +c$ as $t \to -\infty$.