Answers to Selected Problems in Chapters 1 and 2

1. Problem Set 1.4.1

   - 4) a) Two solutions are $x(t) = t^2/4$ and $x(t) \equiv 0$
     b) If $f(x) = \sqrt{x}$ satisfied a Lipschitz condition on $[0, a)$ there would be positive constant L such that $\|\sqrt{x} - 0\| < L\|x\|$ there, or $x > 1/L^2$; since x can approach 0, this is not possible.

2. Problem Set 2.1.1

   - 2) Its clear from the linearity that $Lu = r$ with $u(x) = \sum_{i=1}^{k} u_i(x)$. Since each of the u_i satisfies the initial condition $u_i(x_0) = u_i'(x_0) = 0$, so does their sum $u(x)$. That u is the unique solution of the initial-value problem (2.19) follows from the uniqueness theorem.
   - 6)There are various approaches; here's one. If these three functions were linearly dependent on an interval I then we would have, with $S_k = \sin(kx)$,

     $$\begin{aligned} c_1S_1 + c_2S_2 + c_3S_3 &= 0 \\ -c_1S_1 - 4c_2S_2 - 9c_3S_3 &= 0 \\ c_1S_1 + 16c_2S_2 + 81c_3S_3 &= 0, \end{aligned}$$

     at every point of I, for fixed constants c_1, c_2, c_3. The second pair of equation are obtained from the first equation by differentiating twice and then twice again. The determinant of this system is found to be $204S_1S_2S_3$, and can only vanish at points where $x = m\pi/6$ for certain integers m. But any interval I contains points at which $6x/\pi$ is irrational, and the determinant is nonzero at such points, implying that $c_1 = c_2 = c_3 = 0$.
   - 7) For the first of the identities, $\cos(x + a) = \cos a \cos x - \sin a \sin x$, observe that each side is a solution of the equation $u'' + u = 0$, and that both sides take on the same value at $x = a$, as do their first derivatives. They are therefore the same by the uniqueness theorem.
   - 10) The Wronskian of this pair of solutions is $-2(ad - bc)$, so the condition is that $ad - bc \neq 0$.

- 15) For this equation to be in standard form it is necessary that $p(x)$ not vanish on the interval. Dividing through by p to put it in standard form and writing the equation for the Wronskian, one finds

$$W' = -\frac{p'}{p}W \quad \text{or} \quad W(x) = C/p(x).$$

- 18) The functions u and v satisfy the equations

$$pu' + qu = -u'', \quad pv' + qv = -v''.$$

  These are two linear, algebraic equations for p and q with nonvanishing determinant on $[a, b]$.

3. Problem Set 2.1.2

- 1) A particular integral is $U(x) = x$ and the most general solution is $u = x + c_1 \cos x + c_2 \sin x$. The solution with the given initial data is $u = x - \sin x$.
- 4) Using the formula given in equation (2.25) with the solutions $u_1 = \cos x, u_2 = \sin x$ of the homogeneous problem gives

$$G(x, s) = \cos s \sin x - \sin s \cos x = \sin(x - s)$$

  for the influence function.

4. Problem Set 2.3.2)

- 1) The Wronskian is $W = 2$ and is constant because the coefficient $a_1(x)$ vanishes.