Answers to Selected Problems in Chapters 2 and 3

1. PS 2.3.1

- 2) The solutions $\cos x, \sin x, 1$ have the Wronskian $W = 1$; again it is constant since $p_1(x) \equiv 0$.
- 3) Solutions are $1, x, \exp(-x)$ and $W = \exp(-x)$. The theorem in question gives for the Wronskian $W = C\ \exp(-x)$.
- 7) A particular integral is $U(x) = x$, by inspection. Therefore the most general solution is

$$u(x) = x + c_1 \cos x + c_2 \sin x + c3.$$

- 8) Applying equation (2.34) we find that $G(x, s) = c_1(s) \cos x + c_2(s) \sin x + c_3(s)$. The initial conditions at $x = s$ are

$$c_1 \cos s + c_2 \sin s + c_3 = 0, \quad c_1 \sin s - c_2 \cos x = 0, \quad c_1 \cos s + c_2 \sin s = 1.$$

These are easily solved to find $c_1 = -\cos s, c_2 = -\sin s, c_3 = 1$ so

$$G(x, s) = 1 - \cos(x - s).$$

- 9) It suffices to show that for the homogeneous problem

$$u''' + a_1(x)u'' + a_2(x)u' + a_3(x)u = 0,$$

the only solution for which u, u', and u'' all vanish at the same point x_0 is the solution $u \equiv 0$. Define

$$\sigma(x) = u(x)^2 + u'(x)^2 + u''(x)^2.$$

This vanishes at x_0. Differentiating, eliminating u''' through use of the differential equation, and bounding the coefficients a_1, a_2, a_3 by suitable constants, one finds that $|\sigma'| \leq K\sigma$ for some sufficiently large constant K. Therefore, for $x > x_0$,

$$\sigma(x) \leq K \int_{x_0}^{x} \sigma(s)\, ds$$

implying that $\sigma = 0$ by Gronwall's lemma. This implies that $u \equiv 0$. A similar argument holds for $x < x_0$.

- 11)

$$A(x) = \begin{pmatrix} 0 & 1 \\ -x^4 & -x^2 \end{pmatrix}, \quad r(x) = \begin{pmatrix} 0 \\ 1/(1+x^2) \end{pmatrix}.$$

- 12) With $v(x) = \Phi(x)w(x)$,

$$v'(x) = \Phi'(x)w(x) + \Phi(x)w'(x) = A(x)\Phi(x)w(x)$$

  so the equation reduces to $\Phi(x)w'(x) = r(x)$ or

$$w(x) = \int_{x_0}^{x} \Phi^{-1}(s)r(s)\,ds;$$

  therefore a particular integral is

$$\Phi(x)\int_{x_0}^{x} \Phi^{-1}(s)r(s)\,ds.$$

2. PS 3.2.1

- 12) The characteristic polynomial is $\lambda^2 + 2a\lambda + b$. If $a^2 > b$ the solutions are exponentials $\exp(\lambda_1 x), \exp(\lambda_2 x)$ where $\lambda_{1,2} = -a \pm \sqrt{z^x - b}$. If $a^2 < b$ the solutions are $\exp(-ax)\cos(\omega x), \exp(-ax)\sin(\omega x)$ where $\omega = \sqrt{b - a^2}$. If $a^2 = b$ the solutions are $\exp(-ax), x\exp(-ax)$.
- 13) The characteristic polynomial is $\lambda^3 + 4\lambda^2 + 4\lambda$. Its roots are $\lambda = 0$ and $\lambda = -2$, the latter with multiplicity two. A basis of solutions is therefore

$$u_1 = 1,\ u_2 = e^{-2x},\ u_3 = xe^{-2x}.$$

- 14)The characteristic polynomial is $\lambda^4+2\lambda^2+3$. This is a quadratic in λ^2 with roots $\lambda^2 = -1 \pm i\sqrt{2}$. With $\lambda = a + ib$ one finds $a = \pm 1, b = \pm\sqrt{2}$, providing the four roots. Real basis functions are

$$\exp x \cos(\sqrt{2}x), \exp x \sin(\sqrt{2}x), \exp(-x)\cos(\sqrt{2}x), \exp(-x)\sin(\sqrt{2}x).$$