# Answers to Selected Problems in Chapter 3

1. PS 3.5.1

   - 4) We can read off the characteristic polynomials. For (a) it is $(\lambda+1)^2(\lambda+2) = \lambda^3 + 4\lambda^2 + 5\lambda + 2$ so $a_1, a_2, a_3 = 4, 5, 2$. For (b) it is $\left(\lambda^2+4\right)(\lambda-2) = \lambda^3 - 2\lambda^2 + 4\lambda - 8$ so $a_1, a_2, a_3 = -2, 4, -8$.
   - 5)
     (a) The characteristic polynomial is
     $$p(\lambda) = \lambda^3 - (2\alpha+1)\lambda^2 + \left(\alpha^2+\beta^2+2\alpha\right)\lambda - \left(\alpha^2+\beta^2\right) = 0.$$
     This is satisfied by $\lambda = 1$ so $e^t$ is a solution.

     (b) Since $p$ must contain the factor $\lambda - 1$ it has the form $p(\lambda) = (\lambda-1)\left(\lambda^2 + A\lambda + B\right)$ for some constants $A$ and $B$. Expanding this expression for $p$ and comparing it with the former expression identifies $A = -2\alpha$ and $B = \alpha^2 + \beta^2$. The remaining two roots of the characteristic equation are $\lambda = \alpha \pm i\beta$, so the remaining basis functions are $\exp(\alpha + i\beta)t$ and $\exp(\alpha - i\beta)t$.
   - 6) (recall that in this problem there was a correction: $v_4 = xe^{\overline{\lambda}}$).
     $$u_1 = \frac{1}{2}(v_1+v_2),\ u_2\frac{1}{2i}(v_1-v_2),\ u_3\frac{1}{2}(v_3+v_4),\ u_4\frac{1}{2i}(v_3-v_4)$$
     so
     $$A = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/(2i) & -1(2i) & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/(2i) & -1(2i) \end{pmatrix}.$$
     The determinant is $-1/4$ so this matrix is nonsingular.

2. PS 3.6.1

   - 4) Observe that if $v(x) = u(x+\alpha)$ for constant $\alpha$ then $D^k v(x) = D^k u(x+\alpha)$ for $k = 1, 2, \ldots, n$. Then evaluating $Lu$ at $x + \alpha$ we see in virtue of the fact that the coefficients are independent of $x$ that $Lv = 0$, i.e., that $u(x+\alpha)$ is a solution.
   - 5) A particular integral can be found in the form $u_P = A\cos(\sigma t) + B\sin(\sigma t)$. This leads to
     $$u_p(t) = \frac{1-\sigma^2}{\Delta}\cos(\sigma t) + \frac{2\nu}{\Delta}\sin(\sigma t),\ \Delta = (1-\sigma^2)^2 + 4\nu^2\sigma^2. \quad (1)$$
     The homogeneous equation has the solutions
     $$u_1(t) = e^{-\nu t}\cos(\omega t),\ u_2(t) = e^{-\nu t}\sin(\omega t),\ \omega = \sqrt{1-\nu^2},$$

so the general solution is $u(t) = u_p(t) + c_1 u_1(t) + c_2 u_2(t)$. Applying the initial conditions now gives

$$u(t) = u_p(t) - \frac{1-\sigma^2}{\Delta} u_1(t) - \frac{\nu(1+\sigma^2)}{\Delta} u_2(t).$$

The exponential factors cause $u_1$ and $u_2$ to decay as $t \to \infty$ so the solution tends toward the oscillatory term $u_p(t)$, given by equation (1). It can be written as $u_P = C\cos(\sigma t + \alpha)$ for some amplitude $C$ and phase $\alpha$. It follows from the identity $\cos(x+y) = \cos x \cos y - \sin x \sin y$ that

$$C\cos\alpha = \frac{1-\sigma^2}{\Delta},\ C\sin\alpha = -\frac{2\nu\sigma}{\Delta},$$

so

$$C^2 = \frac{(1-\sigma^2)^2 + 4\nu^2\sigma^2}{\Delta^2} = 1/\Delta,$$

where we have used the formula for $\Delta$ given in equation (1). Therefore

$$C = \frac{1}{\sqrt{\Delta}} = \frac{1}{\sqrt{(1-\sigma^2)^2 + 4\nu^2\sigma^2}}.$$

For fixed $\nu$ this is largest when $\sigma = \pm 1$ and very large indeed when the damping ($\nu$) is small.

- 6) Operating on either side with $D^4$ gives a homogeneous equation, and the four-fold root $\lambda = 0$ does not agree with any of those of the original operator, so a solution can be sought in the form $U = At^3 + Bt^2 + Ct + D$. Substituting this in the equation $LU = t^3$ gives $B = D = 0$ and $A = 1/4, C = 15/8 : U = (1/4)t^3 + (15/8)t$.

3. PS 3.7.1

- 1. For $n = 2$ the characteristic polynomial is

$$p_A(\lambda) = \begin{vmatrix} -\lambda & 1 \\ -a_2 & -a_1 - \lambda \end{vmatrix} = \lambda(\lambda + a_2) + a_1$$

which agrees with the characteristic polynomial of the operator $L$. Assume by induction that

$$p_A(\lambda) = (-1)^k p_L(\lambda) \tag{2}$$

for the system of size $k$. Expand the determinant representing $p_A(\lambda)$ of size $n$ by the first column. This gives

$$-\lambda \begin{vmatrix} -\lambda & 1 & \cdots & 0 \\ 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_2 & -a_3 & \cdots & -a_n - \lambda \end{vmatrix} + (-1)^{n-1}(-a_1) \begin{vmatrix} 1 & 0 & \cdots & 0 \\ -\lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix}.$$

The first term is, by the induction hypothesis,

$$-\lambda(-1)^{n-1}\left(\lambda^{n-1}+a_n\lambda^{n-2}+\ldots+a_2\right)$$

and the second is $(-1)^{n-1}a_1$. Adding these we get the relation (2).

- 3) Note that $\sigma^2=-I$ so

$$\begin{aligned}\exp\sigma &= I+\sigma+\frac{1}{2!}\sigma^2+\frac{1}{3!}\sigma^3+\ldots\\ &= \left(1-\frac{1}{2!}+\frac{1}{4!}-\cdots\right)I+\left(1-\frac{1}{3!}+\frac{1}{5!}-\cdots\right)\sigma\\ &= (\cos 1)\,I+(\sin 1)\,\sigma.\end{aligned}$$

- 6) Writing $A=\lambda I+Z$ where

$$z=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

we find that $Z^2=0$ so

$$A^2=\lambda^2 I+2\lambda Z,\quad A^3=\lambda^3 I+3\lambda^2 Z$$

and so on, so $\exp A=e^{\lambda}(I+Z)$.

- 7) Repeating the previous exercise we find that $\exp(At)=e^{\lambda t}(I+tZ)$. Therefore

$$x(t)=\exp(At)\begin{pmatrix}1\\ -1\end{pmatrix}=\begin{pmatrix}\lambda-1-\lambda t\\ -\lambda\end{pmatrix}.$$

- 8) Using $\exp(At)$ as in the preceding exercise, we find for a particular integral

$$e^{\lambda t}(I+tZ)\int_0^t e^{-\lambda s}(I-sZ)\begin{pmatrix}s\\ 1\end{pmatrix}ds.$$

After some elementary integrations one finds the particular integral

$$U(t)=\frac{e^{\lambda t}-1}{\lambda}\begin{pmatrix}t\\ 1\end{pmatrix},$$

which satisfies the initial condition.