Answers to Selected Problems in Chapter 3

1. PS 3.5.1

- 4) We can read off the characteristic polynomials. For (a) it is
  \((\lambda + 1)^2 (\lambda + 2) = \lambda^3 + 4\lambda^2 + 5\lambda + 2\) so \(a_1, a_2, a_3 = 4, 5, 2\). For (b) it is
  \((\lambda^2 + 4) (\lambda - 2) = \lambda^3 - 2\lambda^2 + 4\lambda - 8\) so \(a_1, a_2, a_3 = -2, 4, -8\).

- 5) (a) The characteristic polynomial is
  \[ p(\lambda) = \lambda^3 - (2a + 1)\lambda^2 + (a^2 + \beta^2 + 2a) \lambda - (a^2 + \beta^2) = 0. \]

  This is satisfied by \(\lambda = 1\) so \(e^t\) is a solution.

  (b) Since \(p\) must contain the factor \(\lambda - 1\) it has the form
  \[ p(\lambda) = (\lambda - 1) (\lambda^2 + A\lambda + B) \]

  for some constants \(A\) and \(B\). Expanding this expression for \(p\) and comparing it with the former expression
  identifies \(A = -2a\) and \(B = a^2 + \beta^2\). The remaining two roots of the characteristic equation are
  \(\lambda = a \pm i\beta\), so the remaining basis functions are \(\exp(\alpha + i\beta)t\) and \(\exp(\alpha - i\beta)t\).

- 6) (recall that in this problem there was a correction: \(v_4 = xe^{\lambda^3}\)).

  \[ u_1 = \frac{1}{2} (v_1 + v_2), \quad u_2 = \frac{1}{2i} (v_1 - v_2), \quad u_3 = \frac{1}{2} (v_3 + v_4), \quad u_4 = \frac{1}{2i} (v_3 - v_4) \]

  so

  \[ A = \begin{pmatrix}
  1/2 & 1/2 & 0 & 0 \\
  1/(2i) & -1/(2i) & 0 & 0 \\
  0 & 0 & 1/2 & 1/2 \\
  0 & 0 & 1/(2i) & -1/(2i)
  \end{pmatrix}. \]

  The determinant is \(-1/4\) so this matrix is nonsingular.

2. PS 3.6.1

- 4) Observe that if \(v(x) = u(x + \alpha)\) for constant \(\alpha\) then \(D^k v(x) = D^k u(x + \alpha)\) for \(k = 1, 2, \ldots, n\). Then evaluating \(Lu\) at \(x + \alpha\) we see

  in virtue of the fact that the coefficients are independent of \(x\) that \(Lv = 0\), i.e., that \(u(x + \alpha)\) is a solution.

- 5) A particular integral can be found in the form \(u_P = A \cos(\sigma t) + B \sin(\sigma t)\). This leads to

  \[ u_p(t) = \frac{1 - \sigma^2}{\Delta} \cos(\sigma t) + \frac{2\nu}{\Delta} \sin(\sigma t), \quad \Delta = (1 - \sigma^2)^2 + 4\nu^2\sigma^2. \quad (1) \]

  The homogeneous equation has the solutions

  \[ u_1(t) = e^{-\nu t} \cos(\omega t), \quad u_2(t) = e^{-\nu t} \sin(\omega t), \quad \omega = \sqrt{1 - \nu^2}, \]

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so the general solution is \( u(t) = u_p(t) + c_1 u_1(t) + c_2 u_2(t) \). Applying the initial conditions now gives

\[
u(t) = u_p(t) - \frac{1-\sigma^2}{\Delta} u_1(t) - \frac{\nu(1+\sigma^2)}{\Delta} u_2(t).
\]

The exponential factors cause \( u_1 \) and \( u_2 \) to decay as \( t \to \infty \) so the solution tends toward the oscillatory term \( u_p(t) \), given by equation (1). It can be written as \( u_p = C \cos(\sigma t + \alpha) \) for some amplitude \( C \) and phase \( \alpha \). It follows from the identity \( \cos(x + y) = \cos x \cos y - \sin x \sin y \) that

\[
C \cos \alpha = \frac{1-\sigma^2}{\Delta}, \quad C \sin \alpha = -\frac{2\nu \sigma}{\Delta},
\]

so

\[
C^2 = \frac{(1-\sigma^2)^2 + 4\nu^2 \sigma^2}{\Delta^2} = 1/\Delta,
\]

where we have used the formula for \( \Delta \) given in equation (1). Therefore

\[
C = \frac{1}{\sqrt{\Delta}} = \sqrt{\frac{1-(\sigma^2)^2 + 4\nu^2 \sigma^2}{\Delta^2}}
\]

For fixed \( \nu \) this is largest when \( \sigma = \pm 1 \) and very large indeed when the damping \( (\nu) \) is small.

6) Operating on either side with \( D^4 \) gives a homogeneous equation, and the four-fold root \( \lambda = 0 \) does not agree with any of those of the original operator, so a solution can be sought in the form \( U = At^3 + Bt^2 + Ct + D \). Substituting this in the equation \( LU = t^3 \) gives \( B = D = 0 \) and \( A = 1/4, C = 15/8 : U = (1/4)t^3 + (15/8)t \).

3. PS 3.7.1

1. For \( n = 2 \) the characteristic polynomial is

\[
p_A(\lambda) = \begin{vmatrix} -\lambda & 1 \\ a_2 & -a_1 - \lambda \\ \end{vmatrix} = \lambda(\lambda + a_2) + a_1
\]

which agrees with the characteristic polynomial of the operator \( L \). Assume by induction that

\[
p_A(\lambda) = (-1)^k p_L(\lambda)
\]

for the system of size \( k \). Expand the determinant representing \( p_A(\lambda) \) of size \( n \) by the first column. This gives

\[
\begin{vmatrix} -\lambda & 1 & \cdots & 0 \\ 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_2 & -a_3 & \cdots & -a_n - \lambda \\ \end{vmatrix} += (-1)^{n-1}(-a_1)
\]

\[
\begin{vmatrix} 1 & 0 & \cdots & 0 \\ -\lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \end{vmatrix}
\]
The first term is, by the induction hypothesis,
\[-\lambda(-1)^{n-1} (\lambda^{n-1} + a_n \lambda^{n-2} + \ldots + a_2)\]
and the second is \((-1)^{n-1}a_1\). Adding these we get the relation (2).

• 3) Note that \(\sigma^2 = -I\) so
\[
\exp \sigma = I + \sigma + \frac{1}{2!} \sigma^2 + \frac{1}{3!} \sigma^3 + \ldots
\]
\[
= \left(1 - \frac{1}{2!} + \frac{1}{4!} - \cdots \right) I + \left(1 - \frac{1}{3!} + \frac{1}{5!} - \cdots \right) \sigma
\]
\[
= (\cos 1) I + (\sin 1) \sigma.
\]

• 6) Writing \(A = \lambda I + Z\) where
\[
z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]
we find that \(Z^2 = 0\) so
\[
A^2 = \lambda^2 I + 2\lambda Z, \quad A^3 = \lambda^3 I + 3\lambda^2 Z
\]
and so on, so \(\exp A = e^{\lambda}(I + Z)\).

• 7) Repeating the previous exercise we find that \(\exp(At) = e^{\lambda t}(I+tZ)\).

Therefore
\[
x(t) = \exp(At) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \lambda - 1 - \lambda t \\ -\lambda \end{pmatrix}.
\]

• 8) Using \(\exp(At)\) as in the preceding exercise, we find for a particular integral
\[
e^{\lambda t}(I + tZ) \int_0^t e^{-\lambda s}(I - sZ) \begin{pmatrix} s \\ 1 \end{pmatrix} ds.
\]
After some elementary integrations one finds the particular integral
\[
U(t) = \frac{e^{\lambda t} - 1}{\lambda} \begin{pmatrix} t \\ 1 \end{pmatrix},
\]
which satisfies the initial condition.