Solutions to Problem Set 5

1. The first three (with the problems restated):

   (a) Consider the second-order, constant-coefficient problem

   $$Lu \equiv D^2 u + a_1 Du + a_2 u = r(x)$$

   with initial values $u(0) = \alpha_0, Du(0) = \alpha_1$. Define the Laplace transform:

   $$f(s) = \int_0^\infty e^{-sx} u(x)\, dx, \tag{1}$$

   where s is a complex number. Assuming that the real part of s may be chosen large enough for all integrals that arise to converge, find the expression for the Laplace transform f.

   Solution: This is based on the observation that, by integration by parts,

   $$\int_0^\infty e^{-st} \dot{u}(t)\, dt = -\dot{u}_0 + s \int_0^\infty e^{-st} u(t)\, dt (= -u_0 + sf(s)).$$

   This formula can be applied successively to get the Laplace transformation of higher derivatives. Applying it to either side of the differential equation now gives

   $$f(s) = \left(\hat{r}(s) + \dot{u}_0 + (a_1 + s)u_0\right)/p(s)$$

   where $p(s) = s^2 + a_1 s + a_2$ is the characteristic polynomial and $\hat{r}$ is the Laplace transform of the function r.

   (b) Find all solutions of the equation

   $$y'(x) = ay(x) + by(c - x) \tag{2}$$

   that exist for all real x. Here a, b, c are real, non-zero constants. For definiteness assume that $a^2 > b^2$.

   Solution: This is not differential equation as normally defined, but a differential-difference equation, since the argument is retarded in the second term on the right. If we put $y_1(x) = y(x)$ and $y_2(x) = y(c-x)$ and observe that $y_2' = -y'(c - x) = -(ay_2(x) + by_1(x))$ then we get the system

   $$y_1' = ay_1 + by_2, \quad y_2' = -by_1 - ay_2.$$

   This can be solved as a system but in this case it is easy to find a second-order equation for $y_1 = y$:

   $$y_1'' - (a^2 - b^2)y_1 = 0.$$

   It follows (with $\omega = \sqrt{a^2 - b^2}$) that

   $$y_1(x) = y(x) = c_1 e^{\omega x} + c_2 e^{-\omega x},$$

where c_1, c_2 are as yet undetermined constants.

This relation is a necessary but not sufficient condition for equation (2) to hold. To determine whether it provides a solution for some values of c_1 and c_2 we substitute it into the original equation obtaining, in view of the linear independence of the functions $\exp(\omega x), \exp(-\omega x)$ the pair of equations

$$(a-\omega)c_1 + be^{-\omega c}c_2 = 0, \quad be^{\omega c}c_1 + (a+\omega)c_2 = 0.$$

The determinant of this system vanishes by virtue of the definition of ω, so there exists a nontrivial choice of c_1 and c_2. For example,

$$c_1 = be^{-\omega c}, \quad c_2 = \omega - a$$

will do.

(c) For the companion matrix (equation 3.65 in the notes) show that whenever λ is an eigenvalue, there is a unique[1] eigenvector belonging to it. By contrast, show by example that for more general matrices, there may be two or more linearly independent eigenvectors belonging to a single eigenvalue.

Solution: If λ is an eigenvalue, the first $n-1$ component equations are

$$x_1 = \lambda x_2, x_2 = \lambda x_3, \ldots, x_{n-1} = \lambda x_n.$$

Therefore, once x_n is specified, all other components are known. This defines the eigenvector uniquely up to scalar multiplication.

On the other hand, suppose the equation is $\dot{x} = Ix$ where I is the $n \times n$ identity matrix. All eigenvalues are equal to one, and there are n linearly independent eigenvectors.

2. PS 4.4.1

Answers to Selected Problems in Chapter 4

- 2) The corresponding series are those for $\exp(-z), \cosh z, \sinh z$ respecitvely.
- 5) $q(z) = -2(1+z)^{-2} = -2+4z-6z^2+\ldots$ with $q_l = 2(l+1)(-1)^{l+1}$. From the recursion formula (4.9) of the text with $w_0 = 1, w_1 = 2$ one finds that $w_2 = 1$ and $w_3 = 0$. The claim is that then $w_k = 0$ for all $k \geq 3$. This can be proved by induction. It's true for 3 and assume it's true for $n = 3, 4, \ldots k+1$. The recursion relation then implies that

$$(k+1)(k+2)w_{k+2} + \{q_2 + 2q_1 + q_0\} = 0.$$

The sum in curly braces vanishes by virtue of the formula for q_l, completing the induction argument. Thus this solution is $w(z) = 1 + 2z + z^2$ as in the example.

[1] As usual with such linear problems, unique up to multiplication by a real or complex constant.

- 6) The recursion formula found in this manner is

$$(k+2)w_{k+2} + 2kw_{k+1} + (k-2)w_k = 0.$$

  With the choices $w_0 = 1, w_1 = 2$ this gives $w_2 = 1$, $w_3 = 0$ and then $w_4 = 0$. Once two consecutive coefficients are zero it is clear from the formula that all subsequent coefficients are also zero,
- 7) Writing the equation in the form $(1+z^2)w'' + w = 0$ leads easily to the recursion formula

$$w_{l+2} = -\frac{l^2 - l + 1}{l^2 + 3l + 2} w_l.$$

- 8) The nearest singularity to the origin occurs where the coefficient $1 + Az + Bz^2 = 0$, or at $z = (-A \pm \sqrt{A^2 - 4B})/2B$. If $A^2 \geq 4B$ the nearer of the two roots is at a distance

$$r = \left(|A| - \sqrt{A^2 - 4B}\right)/2|B|$$

  and if $A^2 < 4B$ the two roots lie at a distance $r = 1$ from the origin. Any power-series solution (in powers of z) has radius of convergence at least r.
- 9) The recursion formula may be written

$$(k+2)(k+1)w_{k+2} = -\left\{Bk^2 + (D-B)k + E\right\} w_k.$$

  The solution may be chosen to be an even function of z by choosing $w_0 \neq 0$ but $w_1 = 0$, for then all coefficients with odd indices vanish. Similarly one may choose an odd solution. Suppose for definiteness an even solution is chosen and suppose that the coefficients B, D, E are such that, for some even integer K, $BK^2 + (D-B)K + E = 0$. Then $W_{K+2} = 0$ and therefore $W_{K+2l} = 0$ for $l = 1, 2, \ldots$. The solution is therefore a polynomial of degree K.
- 11) The power-series solution must satisfy the equation

$$\sum_{)}^{\infty}(l+1)(l+2)w_{l+2}z^l + \sum_{0}^{\infty} w_k z^{k+1} = 0,$$

  so $w_2 = 0$ and the recursion formula is

$$w_{m+3} = \frac{-1}{(m+2)(m+3)} w_m, \quad m = 0, 1, \ldots.$$

  Therefore with $w_0 = 1$ and $w_1 = 0$ the only nonvanishing coefficients are those whose indices are multiples of three:

$$w(z) = 1 - \frac{1}{6}z^3 + \frac{1}{2.3.5.6}z^6 + \ldots.$$

  The radius of convergence is infinite.