Solutions to Problem Set 5

1. The first three (with the problems restated):

(a) Consider the second-order, constant-coefficient problem

\[ Lu \equiv D^2u + a_1Du + a_2u = r(x) \]

with initial values \( u(0) = \alpha_0, Du(0) = \alpha_1 \). Define the Laplace transform:

\[ f(s) = \int_0^\infty e^{-sx}u(x) \, dx, \tag{1} \]

where \( s \) is a complex number. Assuming that the real part of \( s \) may be chosen large enough for all integrals that arise to converge, find the expression for the Laplace transform \( f \).

Solution: This is based on the observation that, by integration by parts,

\[ \int_0^\infty e^{-st} \dot{u}(t) \, dt = -\dot{u}_0 + s \int_0^\infty e^{-st}u(t) \, dt = -u_0 + sf(s). \]

This formula can be applied successively to get the Laplace transformation of higher derivatives. Applying it to either side of the differential equation now gives

\[ f(s) = \left( \hat{r}(s) + \dot{u}_0 + (a_1 + s)u_0 \right) / p(s) \]

where \( p(s) = s^2 + a_1s + a_2 \) is the characteristic polynomial and \( \hat{r} \) is the Laplace transform of the function \( r \).

(b) Find all solutions of the equation

\[ y''(x) = ay(x) + by(c - x) \tag{2} \]

that exist for all real \( x \). Here \( a, b, c \) are real, non-zero constants. For definiteness assume that \( a^2 > b^2 \).

Solution: This is not differential equation as normally defined, but a differential-difference equation, since the argument is retarded in the second term on the right. If we put \( y_1(x) = y(x) \) and \( y_2(x) = y(c - x) \) and observe that \( y_2' = -y_1'(c - x) = -(ay_2(x) + by_1(x)) \) then we get the system

\[ y_1' = ay_1 + by_2, \quad y_2' = -by_1 - ay_2. \]

This can be solved as a system but in this case it is easy to find a second-order equation for \( y_1 = y \):

\[ y_1'' - (a^2 - b^2)y_1 = 0. \]

It follows (with \( \omega = \sqrt{a^2 - b^2} \)) that

\[ y_1(x) = y(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}, \]
where \( c_1, c_2 \) are as yet undetermined constants.

This relation is a necessary but not sufficient condition for equation (2) to hold. To determine whether it provides a solution for some values of \( c_1 \) and \( c_2 \) we substitute it into the original equation obtaining, in view of the linear independence of the functions \( \exp(\omega x), \exp(-\omega x) \) the pair of equations

\[ (a - \omega)c_1 + be^{-\omega c}c_2 = 0, \quad be^{\omega c}c_1 + (a + \omega)c_2 = 0. \]

The determinant of this system vanishes by virtue of the definition of \( \omega \), so there exists a nontrivial choice of \( c_1 \) and \( c_2 \). For example,

\[ c_1 = be^{-\omega c}, \quad c_2 = \omega - a \]

will do.

(c) For the companion matrix (equation 3.65 in the notes) show that whenever \( \lambda \) is an eigenvalue, there is a unique eigenvector belonging to it. By contrast, show by example that for more general matrices, there may be two or more linearly independent eigenvectors belonging to a single eigenvalue.

Solution: If \( \lambda \) is an eigenvalue, the first \( n - 1 \) component equations are

\[ x_1 = \lambda x_2, x_2 = \lambda x_3, \ldots, x_{n-1} = \lambda x_n. \]

Therefore, once \( x_n \) is specified, all other components are known. This defines the eigenvector uniquely up to scalar multiplication.

On the other hand, suppose the equation is \( \dot{x} = Ix \) where \( I \) is the \( n \times n \) identity matrix. All eigenvalues are equal to one, and there are \( n \) linearly independent eigenvectors.

2. PS 4.4.1

Answers to Selected Problems in Chapter 4

• 2) The corresponding series are those for \( \exp(-z), \cosh z, \sinh z \) respectively.

• 5) \( q(z) = -2(1 + z)^{-2} = -2 + 4z - 6z^2 + \ldots \) with \( q_l = 2(l+1)(-1)^{l+1} \).

From the recursion formula (4.9) of the text with \( w_0 = 1, w_1 = 2 \) one finds that \( w_2 = 1 \) and \( w_3 = 0 \). The claim is that then \( w_k = 0 \) for all \( k \geq 3 \). This can be proved by induction. It’s true for 3 and assume it’s true for \( n = 3, 4, \ldots k + 1 \). The recursion relation then implies that

\[ (k + 1)(k + 2)w_{k+2} + \{q_2 + 2q_1 + q_0\} = 0. \]

The sum in curly braces vanishes by virtue of the formula for \( q_l \), completing the induction argument. Thus this solution is \( w(z) = 1 + 2z + z^2 \) as in the example.

1As usual with such linear problems, unique up to multiplication by a real or complex constant.
6) The recursion formula found in this manner is

\[(k + 2)w_{k+2} + 2kw_{k+1} + (k - 2)w_k = 0.\]

With the choices \(w_0 = 1, w_1 = 2\) this gives \(w_2 = 1, w_3 = 0\) and then \(w_4 = 0\). Once two consecutive coefficients are zero it is clear from the formula that all subsequent coefficients are also zero.

7) Writing the equation in the form \((1 + z^2)w'' + w = 0\) leads easily to the recursion formula

\[w_{l+2} = -\frac{l^2 - l + 1}{l^2 + 3l + 2}w_l.\]

8) The nearest singularity to the origin occurs where the coefficient \(1 + Az + Bz^2 = 0\), or at \(z = (-A \pm \sqrt{A^2 - 4B})/2B\). If \(A^2 \geq 4B\) the nearer of the two roots is at a distance

\[r = \frac{|A| - \sqrt{A^2 - 4B}}{2|B|}\]

and if \(A^2 < 4B\) the two roots lie at a distance \(r = 1\) from the origin. Any power-series solution (in powers of \(z\)) has radius of convergence at least \(r\).

9) The recursion formula may be written

\[(k + 2)(k + 1)w_{k+2} = -\left\{Bk^2 + (D - B)k + E\right\}w_k.\]

The solution may be chosen to be an even function of \(z\) by choosing \(w_0 \neq 0\) but \(w_1 = 0\), for then all coefficients with odd indices vanish. Similarly one may choose an odd solution. Suppose for definiteness an even solution is chosen and suppose that the coefficients \(B, D, E\) are such that, for some even integer \(K\), \(BK^2 + (D - B)K + E = 0\). Then \(W_{K+2} = 0\) and therefore \(W_{K+2l} = 0\) for \(l = 1, 2, \ldots\). The solution is therefore a polynomial of degree \(K\).

11) The power-series solution must satisfy the equation

\[\sum_{l=0}^{\infty}(l + 1)(l + 2)w_{l+2}z^l + \sum_{k=0}^{\infty}w_kz^{k+1} = 0,\]

so \(w_2 = 0\) and the recursion formula is

\[w_{m+3} = \frac{-1}{(m + 2)(m + 3)}w_m, \quad m = 0, 1, \ldots\]

Therefore with \(w_0 = 1\) and \(w_1 = 0\) the only nonvanishing coefficients are those whose indices are multiples of three: \(w(z) = 1 - \frac{1}{6}z^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6}z^6 + \ldots\). The radius of convergence is infinite.