

Solutions for Problem Set 6

Answers for problems from Chapter 5 of the text

1. PS 5.1.1

- 1) Since μ_1 is a double root, we must have the relations $\alpha = 1 - 2\mu_1$ and $\beta = \mu_1^2$. With substitution $w(z) = z^{\mu_1} \ln z$ in the expression $Lw = z^2 w'' + \alpha x w' + \beta w$ you find

$$Lw = z^{\mu_1} \left\{ (\mu_1^2 + (\alpha - 1)\mu_1 + \beta) \ln z + 2\mu_1 - 1 + \alpha \right\} = 0.$$

- 3) The equation becomes the constant-coefficient equation $u'' + (\alpha - 1)u' + \beta u = 0$ where $u(t) = w(z)$. Reading off the solutions as $\exp(\mu_1 t)$, $\exp(\mu_2 t)$ or as $\exp(\mu_1 t)$, $t \exp(\mu_1 t)$ provides the solutions of the theorem.

2. PS 5.2.1

- 4) For solutions $w_1 = z^{\mu_1} f_1(z)$ and $w_2 = z^{\mu_2} f_2(z)$ we may assume $f_1(0) = f_2(0) = 1$. Their Wronskian determinant is $(\mu_1 - \mu_2) f_1(z) f_2(z) z^{(\mu_2 + \mu_2 - 1)} + \dots$, where the dots denote terms of higher order in z . This is nonzero near the origin, so the solutions cannot be linearly dependent there.

Alternatively,

With $z = r e^{i\theta}$, under the change $\theta \rightarrow \theta + 2\pi$ we have $z^\mu \rightarrow z^\mu e^{2\pi i \mu}$. The linear relation $c_1 w_1(z) + c_2 w_2(z) = 0$ therefore requires as well that $c_1 e^{2\pi i \mu_1} w_1(z) + c_2 e^{2\pi i \mu_2} w_2(z) = 0$. The determinant of this system for the coefficients c_1, c_2 is $w_1 w_2 (e^{2\pi i \mu_1} - e^{2\pi i \mu_2})$ which does not vanish unless $\mu_2 - \mu_1$ is an integer.

- 5) The indicial equation is $\mu^2 - n^2 = 0$, so $\mu = \pm n$. For $n \geq 0$ an integer, the solution is

$$J_n(z) = \sum_{r=0}^{\infty} \frac{(-1)^r z^{n+2r}}{2^{n+2r} r! (n+r)!}.$$

The indices do differ by an integer.

- 6) The singularities of Legendre's equation occur at ± 1 . to check the one at (say) $z = 1$, introduce the new variable ζ such that $z = 1 + \zeta$. With $w(z) = u(\zeta)$ one finds for Legendre's equation

$$\zeta^2 u'' + P(\zeta)\zeta u' + Q(\zeta)u = 0$$

where $P(\zeta) = 2(1 + \zeta)/(2 + \zeta)$ and $Q(\zeta) = -\lambda\zeta/(2 + \zeta)$. This is a regular singular point since P and Q are holomorphic in a neighborhood of $\zeta = 0$. Since $P_0 = 1$ and $Q_0 = 0$, the indicial equation is $\mu^2 = 0$, so the index is $\mu = 0$ and has multiplicity two. The same result is obtained at the other singular point.

- 7) Substituting $w = f(z) \ln z$ in the equation gives

$$\ln z (f'' + pf' + qf) + z^{-2} (2zf' + (zp - 1)f) = 0.$$

With $z = re^{i\theta}$, letting $\theta \rightarrow \theta + 2\pi$ causes $\ln z \rightarrow \ln z + 2\pi i$ but leaves the terms in the parentheses in the preceding equation unchanged, so we have

$$(\ln z + 2\pi i) (f'' + pf' + qf) + z^{-2} (2zf' + (zp - 1)f) = 0.$$

This together with the earlier equation provides the result.

Answers for Supplemetary Problems for Chapter 5

1. Consider the third-order equation having a regular singular point at the origin:

$$z^3 w''' + a_1(z)z^2 w'' + a_2(z)zw' + a_1(z)w = 0,$$

where a_1, a_2, a_3 are analytic in a domain including the origin. This can be transformed to a system of three first-order equations, by the definitions $w_1 = w, w_2 = zw'_1, w_3 = zw'_2$, of the form $zW' = A(z)W$ where W is the vector with components w_1, w_2, w_3 and A is a three-by-three matrix. Work out A .

Solution: It helps to do a preliminary calculation. Put $D \equiv zd/dz$. The equation for w can be rewritten

$$D^3 w = -a_1 w + (a_2 - a_1 - 2)Dw + (3 - a_1)D^2 w.$$

Since $w_1 = w$, $w_2 = Dw$, $w_3 = D^2w$, the matrix A can now be read off:

$$A(z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & (a_1 - a_2 - 2) & (a_1 - 3) \end{pmatrix}.$$

2. Consider the equation

$$(1 - z)z^2w'' + (z - 4)zw' + 6w = 0.$$

- (a) Verify that this equation has a regular singular point at the origin.
- (b) Find the indicial equation and the indices relative to this point.
- (c) For the index with the greater real part, find the recursion relation for the coefficients in the series solution.

Solution:

- (a) On division by $1 - z$ it has the standard form with

$$P(z) = \frac{z - 4}{1 - z}, \quad Q(z) = \frac{6}{1 - z}.$$

- (b) Since $P_0 = -4$, $Q_0 = 6$ the indicial equation is $\mu^2 - 5\mu + 6$ so the indices are $\mu = 2, 3$.
- (c) $Q(z) = 6 \sum_0^\infty z^k$ so $Q_k = 6$ for $k = 0, 1, 2, \dots$; and one finds $P_0 = -4$ but $P_k = -3$ for $k = 1, 2, \dots$. This gives for the recursion formula

$$I(\mu + n)a_n = \sum_{k=0}^{n-1} \{3(\mu + k) - 6\} a_k,$$

where $I(x) = (x - 3)(x - 2)$. With $\mu = 3$ this becomes

$$n(n + 1)a_n = 3 \sum_{k=0}^{n-1} (k + 1)a_k.$$

(The second solution in this case turns out to be $w_2(z) = z^2$.)

The following two problems relate to singular points at infinity. These are investigated by making the transformation $t = 1/z$ and investigating the singular points at $t = 0$. In each case determine whether the point in question is a point of analyticity, a regular singular point, or an irregular singular point. In the case of a regular singular point, find the indices.

For these put $w(z) = u(t)$ and note that

$$w' = -z^{-2}u' \quad \text{and} \quad w'' = z^{-4}u'' + 2z^{-3}u'.$$

3. The equation $w'' + w = 0$.

Solution: this becomes $t^4u'' + 2t^3u' + u = 0$, which has an irregular singular point at the origin.

4. The equation $z^2w'' + w = 0$

Solution: this becomes $t^2u'' + 2tu' + u = 0$, which has a regular singular point at the origin. This is an Euler equation, with indices $(-1 \pm i\sqrt{3})/2$.