Solutions for Problem Set 6

Answers for problems from Chapter 5 of the text

1. PS 5.1.1

• 1) Since \( \mu_1 \) is a double root, we must have the relations \( \alpha = 1 - 2 \mu_1 \) and \( \beta = \mu_1^2 \). With substitution \( w(z) = z^{\mu_1} \ln z \) in the expression \( Lw = z^2 w'' + \alpha x w' + \beta w \) you find

\[
Lw = z^{\mu_1} \left\{ \left( \mu_1^2 + (\alpha - 1) \mu_1 + \beta \right) \ln z + 2 \mu_1 - 1 + \alpha \right\} = 0.
\]

• 3) The equation becomes the constant-coefficient equation \( u'' + (\alpha - 1) u' + \beta u = 0 \) where \( u(t) = w(z) \). Reading off the solutions as \( \exp(\mu_1 t), \exp(\mu_2 t) \) or as \( t \exp(\mu_1 t) \) provides the solutions of the theorem.

2. PS 5.2.1

• 4) For solutions \( w_1 = z^{\mu_1} f_1(z) \) and \( w_2 = z^{\mu_2} f_2(z) \) we may assume \( f_1(0) = f_2(0) = 1 \). Their Wronskian determinant is \( (\mu_1 - \mu_2) f_1(z) f_2(z) z^{(\mu_2 + \mu_2 - 1)} + \cdots \) where the dots denote terms of higher order in \( z \). This is nonzero near the origin, so the solutions cannot be linearly dependent there.

Alternatively, with \( z = re^{i\theta} \), under the change \( \theta \to \theta + 2\pi \) we have \( z^\mu \to z^\mu e^{2\pi i \mu} \).

The linear relation \( c_1 w_1(z) + c_1 w_2(z) = 0 \) therefore requires as well that \( c_1 e^{2\pi i \mu_1} w_1(z) + c_2 e^{2\pi i \mu_2} w_2(z) = 0 \). The determinant of this system for the coefficients \( c_1, c_2 \) is \( w_1 w_2 (e^{2\pi i \mu_1} - e^{2\pi i \mu_2}) \) which does not vanish unless \( \mu_2 - \mu_1 \) is an integer.

• 5) The indicial equation is \( \mu^2 - n^2 = 0 \), so \( \mu = \pm n \). For \( n \geq 0 \) an integer, the solution is

\[
J_n(z) = \sum_{r=0}^{\infty} \frac{(-1)^r z^{n+2r}}{2^{n+2r} r! (n+r)!}.
\]

The indices do differ by an integer.
6) The singularities of Legendre’s equation occur at ±1. to check the one at (say) \( z = 1 \), introduce the new variable \( \zeta \) such that \( z = 1 + \zeta \). With \( w(z) = u(\zeta) \) one finds for Legendre’s equation

\[
\zeta^2 u'' + P(\zeta)\zeta u' + Q(\zeta)u = 0
\]

where \( P(\zeta) = 2(1 + \zeta)/(2 + \zeta) \) and \( Q(\zeta) = -\lambda \zeta/(2 + \zeta) \). This is a regular singular point since \( P \) and \( Q \) are holomorphic in a neighborhood of \( \zeta = 0 \). Since \( P_0 = 1 \) and \( Q_0 = 0 \), the indicial equation is \( \mu^2 = 0 \), so the index is \( \mu = 0 \) and has multiplicity two. The same result is obtained at the other singular point.

7) Substituting \( w = f(z) \ln z \) in the equation gives

\[
\ln z \left( f'' + pf' + qf \right) + z^{-2} \left( 2zf' + (zp - 1)f \right) = 0.
\]

With \( z = re^{i\theta} \), letting \( \theta \to \theta + 2\pi \) causes \( \ln z \to \ln z + 2\pi i \) but leaves the terms in the parentheses in the preceding equation unchanged, so we have

\[
(\ln z + 2\pi i) \left( f'' + pf' + qf \right) + z^{-2} \left( 2zf' + (zp - 1)f \right) = 0.
\]

This together with the earlier equation provides the result.

Answers for Supplementary Problems for Chapter 5

1. Consider the third-order equation having a regular singular point at the origin:

\[
z^3w''' + a_1(z)z^2w'' + a_2(z)zw' + a_1(z)w = 0,
\]

where \( a_1, a_2, a_3 \) are analytic in a domain including the origin. This can be transformed to a system of three first-order equations, by the definitions \( w_1 = w, w_2 = zw_1', w_3 = zw_2' \), of the form \( zW' = A(z)W \) where \( W \) is the vector with components \( w_1, w_2, w_3 \) and \( A \) is a three-by-three matrix. Work out \( A \).

Solution: It helps to do a preliminary calculation. Put \( D \equiv zd/dz \). The equation for \( w \) can be rewritten

\[
D^3w = -a_1w + (a_2 - a_1 - 2)Dw + (3 - a_1)D^2w.
\]
Since \( w_1 = w, w_2 = Dw, w_3 = D^2w, \) the matrix \( A \) can now be read off:

\[
A(z) = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-a_3 & (a_1 - a_2 - 2) & (a_1 - 3)
\end{pmatrix}.
\]

2. Consider the equation

\[
(1 - z)z^2w'' + (z - 4)zw' + 6w = 0.
\]

(a) Verify that this equation has a regular singular point at the origin.

(b) Find the indicial equation and the indices relative to this point.

(c) For the index with the greater real part, find the recursion relation for the coefficients in the series solution.

Solution:

(a) On division by \( 1 - z \) it has the standard form with

\[
P(z) = \frac{z - 4}{1 - z}, \quad Q(z) = \frac{6}{1 - z}.
\]

(b) Since \( P_0 = -4, Q_0 = 6 \) the indicial equation is \( \mu^2 - 5\mu + 6 \) so the indices are \( \mu = 2, 3 \).

(c) \( Q(z) = 6 \sum_0^\infty z^k \) so \( Q_k = 6 \) for \( k = 0, 1, 2, \ldots \); and one finds \( P_0 = -4 \) but \( P_k = -3 \) for \( k = 1, 2, \ldots \). This gives for the recursion formula

\[
I(\mu + n)a_n = \sum_{k=0}^{n-1} \{3(\mu + k) - 6\} a_k,
\]

where \( I(x) = (x - 3)(x - 2) \). With \( \mu = 3 \) this becomes

\[
n(n + 1)a_n = 3 \sum_{k=0}^{n-1} (k + 1)a_k.
\]

(The second solution in this case turns out to be \( w_2(z) = z^2 \).)
The following two problems relate to singular points at infinity. These are investigated by making the transformation \( t = 1/z \) and investigating the singular points at \( t = 0 \). In each case determine whether the point in question is a point of analyticity, a regular singular point, or an irregular singular point. In the case of a regular singular point, find the indices.

For these put \( w(z) = u(t) \) and note that

\[
-w' = -z^{-2}u' \quad \text{and} \quad w'' = z^{-4}u'' + 2z^{-3}u'.
\]

3. The equation \( w'' + w = 0 \).
   Solution: this becomes \( t^4u'' + 2t^3u' + u = 0 \), which has an irregular singular point at the origin.

4. The equation \( z^2w'' + w = 0 \).
   Solution: this becomes \( t^2u'' + 2tu' + u = 0 \), which has a regular singular point at the origin. This is an Euler equation, with indices \(( -1 \pm i\sqrt{3} )/2 \).