

1. Supplementary Problems for 7th Problem Assignment

These problems concern the system of equations

$$w' = B(z)w \tag{1}$$

where the coefficient matrix B is analytic except for isolated singularities. Recall that a singular point is called *Fuchsian* if B has a pole of order (at most) one there; it is called *regular* if the growth of any solution is at most algebraic there, i.e., for any solution $w(z)$, $|w(z)| \leq K|z|^k$ as $z \rightarrow 0$ for some (in general negative) value of k and some positive constant K .

- (a) Let $z = 0$ be an isolated singularity of B and suppose that $\Phi(z)$ is a fundamental matrix solution of equation (1). Show that *either* Φ fails to be analytic in a neighborhood of $z = 0$ *or* its determinant $|\Phi(z)|$ vanishes there.

The original version of this problem was ambiguously stated (mea culpa).

Solution: A function is analytic at 0 if it has a convergent power-series expansion in some neighborhood of 0. Suppose the assertion is false. Then Φ is analytic at the origin and $|\Phi(0)| \neq 0$. The latter implies there is a neighborhood of the origin where $|\Phi(z)| \neq 0$, and therefore $\Phi^{-1}(z)$ exists and is analytic there. Then $\Phi'(z)\Phi^{-1}(z)$ is likewise analytic in a full neighborhood of the origin. But the latter is $B(z)$ so this is a contradiction.

- (b) Let $n = 1$ and suppose again that the origin is an isolated singularity. Suppose the singular point is regular. Prove

Theorem 0.1 *The singular point is Fuchsian, i.e. the coefficient $B(z)$ has at most a pole of order one at $z = 0$.*

Solution: Any solution has the structure $w(z) = z^\mu f(z)$ where f is analytic and single-valued in a punctured disk, so is given by a Laurent expansion there. The condition of the theorem implies that the Laurent series terminates on the negative side so, by redefining μ if necessary, we may assume that $f(z) = \sum_0^\infty f_k z^k$ with $f_0 \neq 0$. Then

$$\frac{w'}{w} = z^{-1} \frac{\mu f_0 + (\mu + 1)z f_1 + \dots}{f_0 + z f_1 + \dots} = B(z),$$

which has a pole of order (at most) one.

- (c) Suppose 0 is a pole of order $r + 1$ for B :

$$B(z) = z^{-r-1}B_{-r-1} + z^{-r}B_{-r} + \cdots + B_0 + B_z + \cdots$$

Prove

Theorem 0.2 *A necessary condition for 0 to be a regular singular point of equation (1) is that $\text{tr}(B_j) = 0$ for $j = -r-1, -r, \dots, -2$.*

Hint: see Problem 13 of Problem Set 2.3.1 of the text.

Solution: The determinant $v(z) \equiv |\Phi(z)|$ of any fundamental matrix solution Φ satisfies the first-order equation $v' = a(z)v$ where $a(z) = \text{tr}(B(z))$. If Φ is limited by algebraic growth then so also is $|\Phi(z)|$. By the preceding theorem, $\text{tr}(B(z))$ has a pole of order at most one. The conclusion follows from this.

- (d) Suppose that B has first-order poles at the points a_1, a_2, \dots, a_p of the complex plane, and is otherwise analytic and bounded throughout the complex plane. Assume the equation is in fact analytic at the point at infinity. Show that the equation becomes

$$w' = \left(\sum_{j=1}^p (z - a_j)^{-1} B_j \right) w,$$

where the p constant matrices $\{B_j\}$ satisfy the extra condition

$$\sum_{j=1}^p B_j = 0.$$

Solution: Let B_j be the residue of $B(z)$ at a_j :

$$B(z) = (z - a_j)^{-1} B_j + D_j(z)$$

where D_j is analytic in a neighborhood of a_j . The function

$$B(z) - \sum_{j=1}^p (z - a_j)^{-1} B_j$$

is analytic and bounded in the complex plane, and is therefore constant there, by Liouville's theorem:

$$B(z) = C + \sum_{j=1}^p (z - a_j)^{-1} B_j$$

for some matrix of constants C . To investigate the point at ∞ put $t = 1/z$, $u(t) = w(z)$ and consider the equation near $t = 0$. It becomes

$$\frac{du}{dt} = - \left(t^{-2} C + t^{-1} \sum_{j=1}^p (1 - ta_j)^{-1} B_j \right) u.$$

The requirement that the equation be Fuchsian at $t = 0$ requires that $C = 0$. The stronger requirement that there be no singularity there (i.e., that the vector field be analytic) requires that the residue vanish there, i.e., $\sum_{j=1}^p B_j = 0$.

2. PS 6.2.1

- 1) Suppose $y(x)$ is a continuous solution of the integral equation. It is clear that it satisfies the initial condition. Since, furthermore, the integrand $f(s, y(s))$ is continuous, the derivative of the integral with respect to the upper limit exists and is equal to $f(x, y(x))$.
- 4) From the equivalent differential equation, $u(x) = \exp(x^3/3)$.
- 8) Differentiating each side of the u' equation and using the v' equation gives an equation relating u'' to u' , u , and v ; a second use of the original u' equation then eliminates v , giving

$$u'' + p(x)u' + q(x)u = 0,$$

where

$$p(x) = -(a + d + b'/b), \quad q(x) = -(a' + bc - ad - ab'/b).$$

In addition to the differentiability requirements, it is necessary that $b(x)$ not vanish on I .

- 13) The domain may be regarded as unrestricted, so either a solution becomes infinite at the endpoints a, b of a maximal interval, or these numbers may be replaced by $-\infty, +\infty$. Consider the quantity $E = Aw_1^2 + Bw_2^2 + Cw_3^2$ for as yet unspecified constants A, B, C . Then

$$\frac{dE}{dx} = 2(Aw_1w_1' + Bw_2w_2' + Cw_3w_3') = 2w_1w_2w_3(A - B - \mu C).$$

Choosing, e.g., $A = 1 + \mu, B = C = 1$, shows that positive quantity

$$E = (1 + \mu)w_1^2 + w_2^2 + w_3^2$$

is constant on solutions. If the solution were unbounded this quantity would likewise be unbounded, but is constant. Hence solutions exist for all real x .

3. PS 6.4.1

- 1) The v_0 referred to is that mentioned in Theorem 6.3.1: we are to consider u_0 near 1. Then

$$U(x, u_0) - U(x, u_0') = \frac{u_0}{1 - u_0x^2} - \frac{u_0'}{1 - u_0'x^2} = \frac{u_0 - u_0'}{(1 - u_0x^2)(1 - u_0'x^2)}.$$

With $u_0 = 1, 1 - u_0x^2 > 3/4$ on the interval in question so for u_0 near 1 we may assume $1 - u_0x^2 > 1/2$ on this interval. Then

$$|U(x, u_0) - U(x, u_0')| \leq 4|u_0 - u_0'|$$

from which the conclusion follows.

- 2) $\phi(x - x_0, U_0)$ satisfies the differential equation $U' = F(U)$ if $\phi(x, U_0)$ does and satisfies the initial data, so the result follows from the uniqueness theorem.