Solutions to Problem Set 8

1. Supplementary problem: find the volume evolution in phase space for the following systems:

   (a) The Lorenz system

   $$\dot{x} = \sigma(y - x), \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - bz,$$

   where σ, r, b are positive constants.

   (b) Euler's equations of rigid-body dynamics

   $$\dot{w}_1 = w_2 w_3, \quad \dot{w}_2 = -w_3 w_1, \quad \dot{w}_3 = -\mu w_1 w_2,$$

   where μ is a constant.

   (c) Hamilton's canonical equations

   $$\dot{x}_k = \frac{\partial H}{\partial x_{n+k}}, \quad \dot{x}_{n+k} = -\frac{\partial H}{\partial x_k}, k = 1, 2, \ldots, n,$$

   where $H = H(x)$ is a smooth function of the $2n$ variables $x = (x_1, \ldots, x_{2n}$.

   Solution: The volume evolution is given by the divergence of the vector field: $\dot{V} = \int_V \text{div } f\, dV$. For the Lorenz equations this is $\dot{V} = -(\sigma + 1 + b)V$ so V decays exponentially; for the other two systems the volume is constant.

2. PS 6.4.1

   - 4) This is an Euler equation with indices $\pm\beta$ so the general solution is $c_1 x^\beta + c_2 x^{-\beta}$ and the initial data imply that $u(x, \beta) = (1/2)(x^\beta - x^{-\beta})$, on the maximal interval $(0, \infty)$. This is a differentiable function of β with $\partial u/\partial\beta = (\ln x/2)(x^\beta - x^{-\beta})$ for $\beta \neq 0$. For $\beta = 0$ the solution is $u = 1$, and the same conclusions hold.
   - 8) For part a) the solution is $(x_1, x_2) = (te^{-\delta t}, e^{-\delta t})$ and for part b) it is

     $$(x_1, x_2) = \left(\frac{e^{-\mu t} - e^{-\delta t}}{\delta - \mu}, e^{-\mu t}\right).$$

     Taking the limit of this in part c) (and noting the expression for x_1 is the difference quotient whose limit is $te^{-\delta t}$) gives the result.
   - 9) The solution for w is $w = (0, 0, 1)$. The equations for v are

     $$\dot{v}_1 = v_2, \quad \dot{v}_2 = -v_1, \quad \dot{v}_3 = 0.$$

     For general initial data these have the solutions

     $$v_1 = c_1 \cos t + c_2 \sin t, \quad v_2 = -c_1 \sin t + c_2 \cos t, \quad v_3 = c_3$$

     where c_1, c_2, c_3 are constants.

3. PS 7.3.1

- 2) The last function of the vector field is $f_{n+1}(t,x) = 1$, so cannot vanish.
- 3) In polar coordinates r, θ the equations become

$$\dot{r} = -r^2(1-r^2)(4-r^2), \quad \dot{\theta} = 1.$$

These provide circular periodic orbits of the original system at $r = 1$ and $r = 2$. There are no other periodic orbits because $r(t)$ is monotone between these, precluding a return to an earlier value.
- 4) Let $y(t) = x(t+T)$ and note that it satisfies the same equation:

$$\dot{y} = \dot{x}(t+T) = f(x(t+T), t+T) = f(x(t+T), t) = f(y,t).$$

Since by assumption $y(t_1) = x(t_1)$ they are equal for all t by uniqueness.
- 5) Suppose there is a periodic solution with period T. It describes a simple, closed curve γ. Let the region enclosed by γ be A. By Green's theorem in the plane,

$$\int_A \operatorname{div} f \, dA = \int_\gamma f(x) \cdot dx = \int_0^T f(x(t)) \cdot \dot{x}(t) \, dt = \int_0^T \|\dot{x}(t)\|^2 \, dt.$$

If the divergence of f vanishes, this requires that $\dot{x} \equiv 0$.
- 6) Assuming as in the preceding problem that there is a periodic orbit we find that

$$\int_0^T \|\dot{x}\|^2 \, dt = \int_\gamma \nabla\phi(x) \cdot dx = \phi(x(T)) - \phi(x(0)) = 0.$$

- 7) Let $\xi(t)$ be a solution of the initial value problem

$$\dot{\xi} = \xi f(\xi, 0), \quad \xi(0) = c.$$

Then the vector function $(\xi(t), 0)$ is a solution of the given system and, by uniqueness, the only such solution with initial data $(c, 0)$. The line $y = 0$ is therefore an invariant curve. Similarly for the line $x = 0$.

Now suppose a solution is sought with x_0, y_0 in the first quadrant. It can never leave the first quadrant because it would have to cross one of the invariant curves $x = 0$ or $y = 0$, violating uniqueness. Likewise for the other quadrants.