Solutions to Problem Set 9

1. Supplementary Problems: these relate to the conditions (7.6) on the flow $\phi(t,x)$ of a dynamical system.

   (a) Solve the one-dimensional system

   $$\dot{x} = -x^2$$

   explicitly for ϕ and verify condition (iii).
   Solution: The flow is $\phi(t,x) = x/(1+tx)$ so

   $$\begin{aligned} &\phi(s,\phi(t,x)) = \phi(t,x)/\left(1+s\phi(t,x)\right) \\ &= \frac{x/(1+sx)}{1+sx/(1+tx)} = x/\left(1+(s+t)x\right) = \phi(s+t,x). \end{aligned}$$

   (b) Suppose you are given a smooth function ϕ satisfying the conditions (7.6). Show that there is a function $f(y)$ of y only – and find it – such that $\phi(t,x)$ is the solution of the initial-value problem

   $$\dot{y} = f(y), \quad y(0) = x.$$

   Solution: Write $\phi(u,V)$ and differentiate the identity $\phi(s+t,x) = \phi(s,\phi(t,x))$ with respect to s:

   $$\phi_u(s+t,x) = \phi_u(s,\phi(t,x)).$$

   Setting $s=0$ gives $\phi_t(t,x) = \phi_t(0,\phi(t,x)$ or $\dot{y} = f(y)$ with $f(y) = \phi_t(0,y)$.

   (c) First do problem 2 of Problem Set 8.3.1. Then consider the nonlinear system

   $$\dot{x} = y, \quad \dot{y} = -x^3,$$

   which has the same linearization about the origin and answer the same question: is the origin stable or unstable?
   Solution: This system has the constant of the motion $E = (1/4)x^4 + (1/2)y^2$. If x_0, y_0 are chosen so that E is less than $(1/4)\epsilon^4$ and also less than $(1/2)\epsilon^2$ (and this can be done for any small ϵ) then $|x(t)| < \epsilon$ and likewise for $|y(t)|$, so the origin is stable.

   (d) As in the preceding problem, but for the system

   $$\dot{x} = y - x^2, \quad \dot{y} = 0.$$

   Solution: Here $y(t) = y_0$ so $\dot{x} = y_0 - x^2$. If $y_0 < 0$ then $x(t) \le y_0 t$ for all t, which grows without bound as $t \to +\infty$. The origin is therefore unstable.

2. PS 7.4.1

- 1) Solutions: If $y(t) = h(x(t)$ then the condition given follows by differentiating with respect to t and using the differential equations. If the condition holds and one solves the problem

  $$\dot{x} = f(x, h(x)), \quad x(0) = x_0,$$

  the system is satisfied by setting $y(t) = h(x(t)$ and therefore represents the unique solution with $x(0) = x_0$ and $y(0) = h(x_0)$.
- 2) Solution: These are verifications.
- 4) In polar coordinates the equations are $\dot{r} = r(1-r)$ and $\dot{\theta} = 1$. The origin is an equilibrium point and the circle $r = 1$ is a periodic orbit into which nearby orbits spiral, whether starting from the inside or the outside.
- 5) In polar coordinates $\dot{r} = r|1 - r|$ and $\dot{\theta} = 1$. Again the origin is an equilibrium point and the circle $r = 1$ is a periodic orbit but now orbits starting inside the circle spiral into the periodic orbit, orbits starting outside spiral away.

3. PS 8.3.1

   - 2) Solution: The origin is unstable: $y(t) = y_0$ and $x(t) = x_0 + y_0 t$, so solutions with $y_0 \neq 0$ grow without bound.
   - 3) Solution: Linearizing about the origin leads to a system $\dot{\xi} = A\xi$ where

     $$A = \begin{pmatrix} -\sigma & 0 & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}.$$

     The characteristic equation is

     $$p(\lambda) \equiv -(b + \lambda)\left(\lambda^2 + (\sigma + 1)\lambda + \sigma(1 - r)\right) = 0$$

     with solutions $\lambda_1 = -b$ and

     $$\lambda_{2,3} = (1/2)\left(-(\sigma + 1) \pm \sqrt{(\sigma +)^2 - 4\sigma(1 - r)}\right).$$

     The radicand is positive for all values of the parameters (it can be rewritten $(\sigma - a)^2 + 4r$) so all roots are real and they are all negative if $r < 1$. Therefore the origin is stable if this condition holds, but unstable if $r > 1$ since one of the roots is positive in this case.