

Supplementary Problems for 7th Problem Assignment

These problems concern the system of equations

$$w' = B(z)w \tag{1}$$

where the coefficient matrix B is analytic except for isolated singularities. Recall that a singular point is called *Fuchsian* if B has a pole of order (at most) one there; it is called *regular* if the growth of any solution is at most algebraic there, i.e., for any solution $w(z)$, $|w(z)| \leq K|z|^k$ as $z \rightarrow 0$ for some (in general negative) value of k and some positive constant K .

1. Let $z = 0$ be an isolated singularity of B and suppose that $\Phi(z)$ is a fundamental matrix solution of equation (1). Show that *either* Φ is singular at $z = 0$ *or* its determinant $|\Phi(z)|$ vanishes there.
2. Let $n = 1$ and suppose again that the origin is an isolated singularity. Suppose the singular point is regular. Prove

Theorem 0.1 *The singular point is Fuchsian, i.e. the coefficient $B(z)$ has at most a pole of order one at $z = 0$.*

3. Suppose 0 is a pole of order $r + 1$ for B :

$$B(z) = z^{-r-1}B_{-r-1} + z^{-r}B_{-r} + \cdots + B_0 + B_z + \dots$$

Prove

Theorem 0.2 *A necessary condition for 0 to be a regular singular point of equation (1) is that $\text{tr}(B_j) = 0$ for $j = -r - 1, -r, \dots, -2$.*

Hint: see Problem 13 of Problem Set 2.3.1 of the text.

4. Suppose that B has first-order poles at the points a_1, a_2, \dots, a_p of the complex plane, and is otherwise analytic and bounded throughout the complex plane. Assume the equation is *also* Fuchsian at the point at infinity. Show that the equation becomes

$$w' = \left(\sum_{j=1}^p (z - a_j)^{-1} B_j \right) w,$$

where the p constant matrices $\{B_j\}$ satisfy the extra condition

$$\sum_{j=1}^p B_j = 0.$$