Supplemetary Problems for Chapter 3

1. Consider the second-order, constant-coefficient problem

$$Lu \equiv D^2u + a_1 Du + a_2 u = r(x)$$

with initial values $u(0) = \alpha_0, Du(0) = \alpha_1$. Define the Laplace transform:

$$f(s) = \int_0^\infty e^{-sx} u(x)\, dx, \tag{1}$$

where s is a complex number. Assuming that the real part of s may be chosen large enough for all integrals that arise to converge, find the expression for the Laplace transform f.

2. Find all solutions of the equation

$$y'(x) = ay(x) + by(c - x) \tag{2}$$

that exist for all real x. Here a, b, c are real, non-zero constants. For definiteness assume that $a^2 > b^2$.

3. For the companion matrix (equation 3.65 in the notes) show that whenever λ is an eigenvalue, there is a unique[1] eigenvector belonging to it. By contrast, show by example that for more general matrices, there may be two or more linearly independent eigenvectors belonging to a single eigenvalue.

[1] As usual with such linear problems, unique up to multiplication by a real or complex constant.