

Mathematics 273 – Solutions to Midterm Exam –  
February 4, 2010

The problems are equally weighted.

1. Find the solution of the initial-value problem for the first-order equation

$$y' = -xy + 3x, \quad y(0) = 1.$$

Note that  $y_P = 3$  is a particular integral, so  $y(x) = 3 + ce^{-x^2/2}$ . The initial condition is satisfied if  $c = -2$ .

2. Let  $D = d/dx$  and consider the equation

$$(D - 1)^3 u \equiv u''' - 3u'' + 3u' - u = 0. \quad (1)$$

- (a) Find a linearly independent set of solutions.  
 $e^x, xe^x, x^2e^x$ .

- (b) Find any particular integral of the equation

$$(D - 1)^3 u = e^x.$$

Since  $(D - 1)e^x = 0$ , any solution must satisfy  $(D - 1)^4 u = 0$  and therefore be a linear combination of the three given above plus  $x^3e^x$ . We can ignore solutions of the homogeneous equation and seek a particular integral in the form  $u_P(x) = Ax^3e^x$ . Substitution in equation (1) then shows that  $A = 1/6$ .

3. The equation

$$w'' - zw = 0$$

is to hold for  $w$  in a domain of the complex plane including the origin. We seek solutions in the form of power series

$$w(z) = w_0 + w_1z + w_2z^2 + \cdots w_kz^k + \cdots = \sum_{k=0}^{\infty} w_kz^k.$$

- (a) Obtain the recursion relation for the coefficients  $\{w_k\}$ .  
 $w_{k+3} = w_k/(k+2)(k+3)$ .

- (b) What is the radius of convergence of this power series?

The radius of convergence of any power-series solution is infinite since that is true of the coefficients of the equation.

4. Consider the initial-value problem on the entire real line  $R$ :

$$u'' + p(x)u' + q(x)u = 0, \quad u(0) = 0, \quad u'(0) = 1. \quad (2)$$

Here  $p$  and  $q$  are continuous, real-valued functions, and

$$q(x) < 0 \text{ for all values of } x \in R.$$

Consider the following assertion:

The solution  $u$  cannot vanish at any point of  $R$  other than the origin.

If this assertion is true, prove it. If it is false, find a specific choice of  $p$  and  $q$  and a solution of the initial-value problem providing a counterexample.

The assertion is true. Since  $u'(0) = 1$  there is an interval to the right of the origin where  $u$  is positive. Suppose there is positive zero  $x_1$ . There must then be a least positive zero, which we can continue to call  $x_1$ . Then  $u > 0$  on  $(0, x_1)$  and therefore has a positive maximum at some point  $x_*$  of that interval. At  $x_*$  we must have

$$u(x_*) > 0, \quad u'(x_*) = 0, \quad u''(x_*) \leq 0.$$

But according to the differential equation with  $u'(x_*) = 0$ ,

$$u''(x_*) = -q(x_*)u(x_*) > 0.$$

This contradiction shows there is no positive zero. A similar analysis shows that there can be no negative zero.