Supplementary Problems for Chapter 2

1. Let $u_1, u_2, \ldots, u_n$ be continuous, real-valued functions on an interval $[a, b]$ of the real axis. Show that a necessary and sufficient condition for these functions to be linearly dependent there is that the Gram matrix $A = (a_{ij})$ be singular (for the Gram matrix, $a_{ij} = \int_a^b u_i(x)u_j(x) \, dx$).

Suppose they are linearly dependent there. Then there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1u_1(x) + c_2u_2(x) + \ldots + c_nu_n(x) = 0$$

on $[a, b]$. Multiplying by $u_i(x)$ and integrating from $a$ to $b$ gives

$$\sum_{k=1}^n a_{ik}c_k = 0, \quad \text{or} \quad Ac = 0,$$

where $A$ is the Gram matrix. Since $c$ is not the zero vector this shows that $A$ is singular. Next suppose that $A$ is singular and put

$$v(x) = \sum_{k=1}^n c_k u_k(x)$$

where $c_1, c_2, \ldots, c_n$ are constants to be chosen. Observe that

$$\int_a^b v(x)^2 \, dx = \sum_{k=1}^n \sum_{j=1}^n c_k c_j a_{kj}.$$

Since $A$ is singular the constants $c_1, c_2, \ldots, c_n$ can be chosen (not all zero) so that $Ac = 0$, i.e., $\sum_{j=1}^n a_{kj}c_j = 0$ for $k = 1, 2, \ldots, n$. Thus the integral of $v^2$ vanishes. For a continuous function this is possible only if $v$ vanishes identically. Thus for the choice of constants as above $\sum c_k u_k(x)$ vanishes on $[a, b]$.

2. Consider the system of $n$ equations

$$\frac{dy}{dt} = A(t)y$$

and suppose the matrix $A$ is upper-triangular: $a_{ij}(t) = 0$ if $i > j$.

Explain how the solution of this equation can be obtained through the
solution of \( n \) first-order equations (do not try to give the complete formula for the solution).

The last equation is
\[
\frac{dy_n}{dt} = a_{nn}(t)y_n,
\]
which can be solved in the form
\[
y_n(t) = c_n \exp\{\int_{t_0}^{t} a_{nn}(s) \, ds\}.
\]

The next to last equation is then
\[
\frac{dy_{n-1}}{dt} = a_{(n-1)(n-1)}(t)y_{n-1} + a_{(n-1)n}(t)y_n(t).
\]
Since \( y_n(t) \) is known from the first step, this is an inhomogenous, first-order equation which can be solved by a known formula, giving \( y_{n-1} \) as an explicit function of \( t \). The remaining equations, from \( n-2 \) up to 1 can be solved successively in this manner.

Answers to Selected Problems in Chapters 2 and 3
1. PS 2.1.2
   - 4) The solutions of the homogeneous problem are \( \cos x \) and \( \sin x \) with Wronskian one, so the influence function is
   \[
   G(x, s) = \cos s \sin x - \sin s \cos x = \sin(x - s).
   \]

2. PS 2.3.1
   - 4) Solutions are 1, \( x \), \( \cos x \), \( \sin x \) with Wronskian = 1.
   - 6) Applying the formula to the Wronskian gives
   \[
   \frac{dW}{dx} = \frac{d}{dx} \begin{vmatrix}
   u_1 & \cdots & u_n \\
   \vdots & \ddots & \vdots \\
   u_1^{(n)} & \cdots & u_n^{(n)}
   \end{vmatrix}
   \]
   since each of the other determinants given by the formula has two rows that are equal. In the bottom row above substitute the expressions for \( u_k^{(n)} \) from the differential equation. This can be expanded into \( n \) terms of which \( n-1 \) again have rows that are proportional to each other. The remaining term is \(-a_1(x)W(x)\) giving the formula in question.
12) With $v(x) = \Phi(x)w(x)$,

$$v'(x) = \Phi'(x)w(x) + \Phi(x)w'(x) = A(x)\Phi(x)w(x) + \Phi(x)w'(x)$$

so the equation reduces to $\Phi(x)w'(x) = r(x)$ or

$$w(x) = \int_{x_0}^{x} \Phi^{-1}(s)r(s) \, ds;$$

therefore a particular integral is

$$\Phi(x)\int_{x_0}^{x} \Phi^{-1}(s)r(s) \, ds.$$

13) Write the fundamental matrix solution in the form

$$\Phi(x) = \begin{pmatrix} u^{(1)} & u^{(2)} & \cdots & u^{(n)} \\ u^{(1)}_2 & u^{(2)}_2 & \cdots & u^{(n)}_2 \\ \vdots & \vdots & \ddots & \vdots \\ u^{(1)}_n & u^{(2)}_n & \cdots & u^{(n)}_n \end{pmatrix}$$

in terms of $n$ linearly independent column vectors $\{u^{(j)}\}$, $j = 1, 2, \ldots, n$. Then differentiating the determinant $|\Phi(x)|$ leads to $n$ determinants:

$$\frac{d|\Phi(x)|}{dx} = \begin{vmatrix} u^{(1)'}_1 & u^{(2)'}_1 & \cdots & u^{(n)'}_1 \\ u^{(1)'}_2 & u^{(2)'}_2 & \cdots & u^{(n)'}_2 \\ \vdots & \vdots & \ddots & \vdots \\ u^{(1)'}_n & u^{(2)'}_n & \cdots & u^{(n)'}_n \end{vmatrix} + \text{ETC}$$

where ETC means the sum of the remaining $n - 1$ determinants. In the determinant above, using the differential equations

$$u^{(1)'}_1 = \sum_{j=1}^{n} a_{ij}u^{(1)}_j$$

in each of the terms of the first row leads, after noting that it can be written as the sum of $n$ determinants of which $n - 1$ vanish, to the formula

$$\frac{d|\Phi(x)|}{dx} = a_{11}(x)|\Phi(x)| + \text{ETC}.$$
Treating the remaining terms in the sum ETC in the same way leads to

$$\frac{|\Phi(x)|}{dx} = a(x)|\Phi(x)|$$

where \(a(x) = (\text{Tr}A(x)) = \sum_{j=1}^n a_{jj}(x)\).

3. PS 3.2.1

• 12) The characteristic polynomial is \(\lambda^2 + 2a\lambda + b\). If \(a^2 > b\) the solutions are exponentials \(\exp(\lambda_1 x), \exp(\lambda_2 x)\) where \(\lambda_{1,2} = -a \pm \sqrt{a^2 - b}\). If \(a^2 < b\) the solutions are \(\exp(-ax) \cos(\omega x), \exp(-ax) \sin(\omega x)\) where \(\omega = \sqrt{b - a^2}\). If \(a^2 = b\) the solutions are \(\exp(-ax), x \exp(-ax)\).

• 13) The characteristic polynomial is \(\lambda^3 + 4\lambda^2 + 4\lambda\). Its roots are \(\lambda = 0\) and \(\lambda = -2\), the latter with multiplicity two. A basis of solutions is therefore

\[u_1 = 1, \quad u_2 = e^{-2x}, \quad u_3 = xe^{-2x}\.\]

• 14) The characteristic polynomial is \(\lambda^4 + 2\lambda^2 + 3\). This is a quadratic in \(\lambda^2\) with roots \(\lambda^2 = -1 \pm i\sqrt{2}\). With \(\lambda = a + ib\) one finds \(a = \pm 1, b = \pm \sqrt{2}\), providing the four roots. Real basis functions are

\[\exp x \cos(\sqrt{2}x), \exp x \sin(\sqrt{2}x), \exp(-x) \cos(\sqrt{2}x), \exp(-x) \sin(\sqrt{2}x)\.\]