Solutions to Problem Set 8

• PS 7.3.1
  
  - 4) Let \( y(t) = x(t+T) \) and note that it satisfies the same equation:
    \[
    \dot{y} = \dot{x}(t+T) = f(x(t+T), t+T) = f(x(t+T), t) = f(y, t).
    \]
    Since by assumption \( y(t_1) = x(t_1) \) they are equal for all \( t \) by uniqueness.
  
  - 5) See the second of the supplementary problems below.
  
  - 7) Let \( \xi(t) \) be a solution of the initial value problem
    \[
    \dot{\xi} = \xi f(\xi, 0), \quad \xi(0) = c.
    \]
    Then the vector function \((\xi(t), 0)\) is a solution of the given system and, by uniqueness, the only such solution with initial data \((c, 0)\).
  
  - 8) To see that the nonwandering set \( N \) is closed for some flow \( \phi(t,.) \), consider a sequence \( \{p_k\} \) of points of \( N \) converging to a point \( p_* \). We need to show that \( p_* \in N \), i.e., we choose any neighborhood \( B_* \) of \( p_* \) and any time \( T_* > 0 \), and we need to show that there is a point \( q \in B_* \) such that \( \phi(t, q) \in B_* \) for some \( t > T_* \). Choose a point \( p_k \) of the sequence lying in \( B_* \) and choose a small enough neighborhood \( b \) of \( p_k \) that \( b \subset B_* \). Since \( \phi(t, p_k) \) returns to \( b \) and therefore to \( B_* \) for some \( t > T_* \), this shows that \( p_* \in N \), i.e., \( N \) is closed.
  
  To see that \( N \) is positively invariant, let \( p_0 \in N \) and consider, for arbitrary \( \tau > 0 \), \( p_1 = \phi(\tau, p_0) \). Let \( B_1 \) be an arbitrary neighborhood of \( p_1 \) and \( T_1 > 0 \). We need to show that there is a point \( q_1 \) of \( B_1 \) such that \( r_1 = \phi(t_1, q_1) \in B_1 \) for \( t_1 > T_1 \). To see this consider the image of the neighborhood \( B_1 \) under the mapping \( \phi(-\tau,.) \). This is a neighborhood \( B_0 \) of \( p_0 \). We can therefore find \( q_0 \in B_0 \) such that \( r_0 = \phi(t_0, q_0) \in B_0 \) for \( t_0 > T_1 \). Then with \( t_1 = t_0 \) we have the corresponding points \( q_1, r_1 \) as described above.
• PS 7.4.1

1) Solutions: If \( y(t) = h(x(t)) \) then the condition given follows by differentiating with respect to \( t \) and using the differential equations. If the condition holds and one solves the problem

\[
\dot{x} = f(x, h(x)), \quad x(0) = x_0,
\]

the system is satisfied by setting \( y(t) = h(x(t)) \) and therefore represents the unique solution with \( x(0) = x_0 \) and \( y(0) = h(x_0) \).

3) If \( \dot{x} = f(x, y), \quad \dot{y} = g(x, y) \) is defined in \( \mathbb{R}^2 \), then this system is Hamiltonian if \( \partial f / \partial x + \partial g / \partial y = 0 \), and this is true for the system in question if \( R = 0 \). To find \( H \) note that

\[
\frac{\partial H}{\partial x} = x^2 + y
\]

so \( H(x, y) = x^2y + (1/2)y^2 + k(x) \). We determine the function \( k(x) \) from the other equation

\[
-\frac{\partial H}{\partial x} = -2xy - k'(x) = -2(1 + y)x
\]

implying that \( k' = 2x \). Therefore

\[
H(x, y) = x^2y + (1/2)y^2 + x^2.
\]

4) In polar coordinates the equations are \( \dot{r} = r(1 - r) \) and \( \dot{\theta} = 1 \). The origin is an equilibrium point and the circle \( r = 1 \) is a periodic orbit into which nearby orbits spiral, whether starting from the inside or the outside.

5) In polar coordinates \( \dot{r} = r|1 - r| \) and \( \dot{\theta} = 1 \). Again the origin is an equilibrium point and the circle \( r = 1 \) is a periodic orbit but now orbits starting inside the circle spiral into the periodic orbit, orbits starting outside spiral away.

Solutions for Supplementary Problems for Chapter 7

1. Consider the system

\[
\dot{x} = -y + x(r^4 - 3r^2 + 1), \quad \dot{y} = x + y(r^4 - 3r^2 + 1),
\]

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where $r^2 = x^2 + y^2$. Show that the only equilibrium point is at the origin, and that $\dot{r} > 0$ if $r = 3$ and $\dot{r} < 0$ if $r = 1$. Infer from these that there is a periodic orbit in the annular region $1 < r < 3$.

Since

$$x \dot{x} + y \dot{y} = 2r \dot{r} = r^2(r^4 - 2r^2 + 1)$$

a necessary condition for an equilibrium point is that the right-hand side vanish. This is true at the origin, which is an equilibrium point. It is also zero for values of $r$ at which the other factor vanishes but at those points $\dot{x} = y$ and $\dot{y} = -x$ so these are not equilibrium points. The formula above can be used to show that $\dot{r} > 0$ at $r = 3$ and that $\dot{r} < 0$ at $r = 1$. Thus, integrating backwards in time, orbits are confined between these two circles and therefore have an $\alpha$-limit set, which is a periodic orbit by the Poincaré-Bendixson theory.

2. Suppose the system of equations

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2)$$

is given in a simply-connected domain $D$ of the plane, and the functions $f_1$ and $f_2$ are $C^1$ there.

(a) Suppose there is a real-valued, $C^1$ function $m(x_1, x_2)$ such that $\text{div}(mf)$ does not vanish identically in $D$ and does not change sign there. Show that there can be no periodic orbit in $D$.

(b) Suppose $f_1 = x_2$ and $f_2 = -ax_1 - bx_2 + cx^2_1 + dx^2_2$. Use the function $m = b \exp\{-2dx_1\}$ to infer that there are no periodic orbits of this system in $\mathbb{R}^2$.

(a) By assumption, if there is a periodic orbit $J$ of period $T$, enclosing the region $D$ then, by Green’s theorem in the plane,

$$0 \neq \int_D \text{div}(mf) \, dx_1 dx_2 = \oint_J (-mf_2 \, dx_1 + mf_1 \, dx_2)$$

$$= \int_0^T m (-f_2 \dot{x}_1 + f_1 \dot{x}_2) \, dt = 0$$

because $\dot{x}_1 = f_1$, $\dot{x}_2 = f_2$.

(b) Calculating $\text{div}(mf)$ gives $-b^2 \exp(-2dx_1)$, which is of one sign.