Solutions for Problem Set 9

1. Problem Set 8.3.1

   - 4) This problem is not accurately stated: the supremum is taken not only over time but also over initial data $x(0) \neq 0$. With this understanding, the solution is as follows:
     The solution of the equations with initial data (c_1, c_2) is
     $$x_1 = c_1 e^{-\delta t} + c_2 t e^{-\delta t},\ x_2 = c_2 e^{-\delta t}.$$
     If $c_1 = 0$ and $t = 1/\delta$, $x_1 = c_1/\delta e$ and therefore $\|x\| > |c_1|/\delta e$. Since $c_1 = \|x(0)\|$ this shows that, for $t = 1/\delta$ the amplification factor exceeds $1/\delta e$ and therefore its supremum exceeds this.
   - 7)
     - Equation (8.16): this follows immediately from the definition of the matrix exponential.
     - Equation (8.17): the issue is to show that $\exp(\lambda I + Z) = e^{\lambda} \exp Z$. Since I commutes with any matrix and for any commuting matrices A and B we have $\exp(A + B) = \exp(A)\exp(B)$, the matrix in question is $\exp(\lambda I)\exp(Z)$. The first of these is the product of the scalar e^{λ} with the unit matrix, and this gives the formula.
     - Equation (8.18): This is the outcome of the formula
       $$e^{Zt} = \sum_{k=0}^{\infty} \frac{t^k}{k!} Z^k = \sum_{k=0}^{m-1} \frac{t^k}{k!} Z^k$$
       where the last form of the expression is due to the nilpotency of Z. Since
       $$Z = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},\ Z^2 = \begin{pmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
       and so on, with the secondary diagonal of ones moving upward toward the upper righthand corner with each futher application of Z, this leads to equation (8.18).

- 12) Let R be as in the hint. Then $R^{-1} = \operatorname{diag}(\nu^m, \nu^{m-1}, \ldots, \nu)$. Applying the similarity transformation $R^{-1}ZR$ provides the required formula.

2. Problem set 8.5.1

- 2) The equiibrium conditions are

$$\sigma(y-x) = 0,\ rx - y - xz = 0,\ xy - bz = 0.$$

Since σ and b are positive, we have $y = x$. The second equation gives $x(r-1-z) = 0$ and we may assume $x \neq 0$ since otherwise we are led back to the origin $(0,0,0)$. Therefore $z = r-1$ and the last equation gives $x^2 = b(r-1)$, providing new equilibrium solutions provided $r > 1$. There are two, depending on the sign chosen for $x_\pm = \sqrt{b(r-1)}$:

$$x = x_\pm,\ y = x,\ z = (r-1). \tag{1}$$

- 3) The Jacobian matrix of the vector field is

$$Df = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}.$$

Evaluating x, y, z at the equilibrium point (1) and evaluating the determinant $|Df - \lambda I|$ we find (up to a minus sign)

$$p(\lambda) = \lambda^3 + (1+\sigma+b)\lambda^2 + b(\sigma+r)\lambda + 2b(r-1).$$

- 4) If $r = 1$ the constant term above vanishes and $\lambda = 0$ is a root. The others are

$$\lambda_\pm = (1/2)\left\{-(1+\sigma+b) \pm \sqrt{(1+\sigma-b)^2}\right) = -b, -(1+\sigma).$$

These are both negative and must remain so for r near 1. Applying the perturbation method to the remaining root gives $\lambda = \epsilon\lambda_1 + \cdots$ with $\lambda_1 = -\frac{2\sigma}{\sigma+1}$. Thus all three roots are negative for $r-1$ positive and sufficiently small.

- 5) The equations are $\dot{x} = x(a - by)$, $\dot{y} = -y(c - dx)$ where the constants a, b, c, d are positive. The equilibrium solutions are $(x, y) = (0, 0)$ and $(x, y) = (c/d, a/b)$. The Jacobian matrix is

$$Df = \begin{pmatrix} a - by & -bx \\ dy & dx - c \end{pmatrix}.$$

Evaluated at $(x, y) = (0, 0)$ this give $Df = \mathrm{diag}(a, -c)$ showing the origin to be unstable. Evaluated at the other equilibrium point one finds

$$Df = \begin{pmatrix} 0 & -bc/d \\ da\ b & 0 \end{pmatrix}$$

so its characteristic polynomial is $\lambda^2 + ac$, giving purely imaginary roots $\pm i\sqrt{ac}$. This is linearly stable but this is a case in which the linear theory is indecisive.

- 6) Following the hint one finds the function

$$E(x, y) = dx + by - c \ln x - a \ln y.$$

- 8) Setting the time derivatives equal to zero gives $\omega_1 \omega_2 = 0$, $\omega_2 \omega_3 = 0$, $\omega_3 \omega_1 = 0$. If $\omega_1 \neq 0$ then $\omega_2 = \omega_3 = 0$, and so on, by cyclic permutation.

- 9) Evaluation of the characteristic polynomial gives

$$p(\lambda) = -\lambda \left(\lambda^2 - \omega_1^2 \frac{(I_3 - I_1)(I_1 - I_2)}{I_2 I_3} \right).$$

If $I_2 < I_1 < I_3$ or if $I_3 < I_1 < I_2$ this gives a positive root and the equilibrium is unstable. In the other cases it gives one root $\lambda = 0$ and a pair that is pure imaginary, so the basic theorems are indecisive. We may reason as follows.

Let $\omega = (c_1, 0, 0)$ represent the equilibrium solution. It is an exercise to check that

$$E(\omega) = (1/2) \sum_{k=1}^{3} I_k \omega_k^2 \text{ and } M(\omega) = (1/2) \sum_{k=1}^{3} I_k^2 \omega_k^2$$

are constant on orbits. Supposing that I_1 is the greatest axis and forming the difference $I_1 E - M$ gives

$$(I_1 - I_2) I_2 \omega_2^2 + (I_1 - I_3) I_3 \omega_3^2 = \Delta,$$

where the constant of the motion Δ can be made as small as we please by choosing the initial values of ω_2 and ω_3 sufficiently near zero. Now reconsidering the function E, we see that $I_1\omega_1$ can be made to remain as close as we please to its equilibrium value c_1 by taking the initial value for ω_1 sufficiently close to c_1. The case when I_1 is the least axis can be treated similarly.