Supplementary Problems for Chapter 7

1. Consider the system

$$\dot{x} = -y + x(r^4 - 3r^2 + 1), \quad \dot{y} = x + y(r^4 - 3r^2 + 1),$$

where $r^2 = x^2 + y^2$. Show that the only equilibrium point is at the origin, and that $\dot{r} > 0$ if $r = 3$ and $\dot{r} < 0$ if $r = 1$. Infer from these that there is a periodic orbit in the annular region $1 < r < 3$.

2. Suppose the system of equations

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2)$$

is given in a simply-connected domain D of the plane, and the functions f_1 and f_2 are C^1 there.

   (a) Suppose there is a real-valued, C^1 function $m(x_1, x_2)$ such that $\operatorname{div}(mf)$ does not vanish identically in D and does not change sign there. Show that there can be no periodic orbit in D.

   (b) Suppose $f_1 = x_2$ and $f_2 = -ax_1 - bx_2 + cx_1^2 + dx_2^2$. Use the function $m = b \exp\{-2dx_1\}$ to infer that there are no periodic orbits of this system in R^2.