The problems are weighted equally

1. Find the solutions of the initial-value problems on \([0, \infty)\):

   (a) \( y' = \frac{x^2}{1+x} y, \ y(0) = 1 \).

   Solution: \( y = \exp \left( \int_0^x \frac{s^2}{1+s^3} \, ds \right) = (1 + x^3)^{1/3} \).

   (b) \( y' = \frac{x^2}{1+x} y^{-1}, \ y(0) = 1 \).

   Solution: \( y y' = \frac{d}{dx} \left( \frac{1}{3} \ln(1 + x^3) \right) \) so \( y = \left( 1 + \frac{2}{3} \ln(1 + x^3) \right)^{1/2} \),
   taking the positive square root so that \( y(1) = +1 \).

2. Consider the second-order differential equation

   \[ u'' + p(x)u' + q(x)u = 0, \quad (1) \]

   where \( p \) and \( q \) are bounded, continuous functions on the entire real axis \( \mathbb{R} \). Suppose that \( q(x) < 0 \) there. Show that a solution \( u \) cannot have more than one zero on \( \mathbb{R} \). Can it have one?

   Solution: We assume that \( u \) is a solution (not identically zero) that has two or more zeros, and let \( x_1 \) and \( x_2 \) be consecutive zeros, i.e., \( u \) does not vanish on the interval \((x_1, x_2)\). For definiteness suppose \( u > 0 \) there. Then \( u \) must have a positive maximum at some point \( x_* \) in this interval. Therefore \( u'(x_*) = 0 \) whereas \( u''(x_*) \leq 0 \). Therefore, according to the differential equation,

   \[ u''(x_*) + q(x_*)u(x_*) = 0. \]

   The second term is strictly negative whereas the first is negative or zero. This is a contradiction, so there cannot be more than one zero.

   There can be one since the equation can be solved with initial data \( u(a) = 0, u'(a) = b \neq 0 \), giving a nontrivial solution with one zero.

3. Let \( Lu \equiv u'' - x^{-1}u' \) for \( x \geq 1 \).
(a) Find a basis of solutions of the homogeneous equation \( Lu = 0 \). 
Solution: \( u_1(x) = 1 \) is a solution by inspection. Putting \( v = u' \) we find for the equation \( v' = x^{-1}v \) the solution \( v = x \) and therefore a second solution is \( u = (1/2)x^2 \) so we may choose \( u_2(x) = x^2 \) as a second, linearly independent solution.

(b) Find the influence function providing a particular integral of the differential equation \( Lu = r(x) \).
Solution: The influence function is obtained by solving the initial-value problem

\[ LG = 0, \quad x > \xi \quad \text{with initial data} \quad G = 0 \quad \text{and} \quad \frac{\partial G}{\partial x} = 1 \quad \text{at} \quad x = \xi. \]

Therefore \( G(x, \xi) = c_1u_1(x) + c_2u_2(x) = c_1 + c_2x^2 \) in the present case. Applying the initial data leads to: \( c_1 = -(1/2)\xi, c_2 = (1/2\xi) \) so

\[ G(x, \xi) = -(1/2)\xi + (1/2)(x^2/\xi). \]

4. Find real bases of solutions for the following equations:

(a) \( u'' + 2u' + 5u = 0 \).
Solution: The roots of the characteristic equation are \( \lambda = -1 \pm 2i \) so real solutions are

\[ u_1 = e^{-x} \cos(2x), \quad u_2 = e^{-x} \sin(2x). \]

(b) \( u'' + 2u' + u = 0 \).
Solution: The roots are \( \lambda = -1 \) with multiplicity two, so the real solutions are

\[ u_1 = e^{-x}, \quad u_2 = xe^{-x}. \]

(c) \( u'''' + 2u'' - 2u' - 4u = 0 \) (Hint: \( \exp(-2x) \) is a solution.)
Solution: The characteristic polynomial \( p(\lambda) \) has a root \( \lambda = -2 \) according to the hint. This implies that \( p(\lambda) = (\lambda + 2)(\lambda^2 - 2) \) giving \( \lambda = \pm \sqrt{2} \) for the remaining two roots. A real basis is therefore

\[ u_1 = \exp(-2x), \quad u_2 = \exp(\sqrt{2}x), \quad u_3 = \exp(-\sqrt{2}x). \]
5. Consider the system of equations
\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = x_2. \quad (2) \]

(a) Find the most general solution of this system.
Solution: \( x_2 = c_2 e^t, \ x_1 = c_1 + c_2 e^t \) where \( c_2 = x_2(0) \) and \( c_1 = x_1(0) - x_2(0) \).

(b) Write the matrix \( A \) expressing this system in the vector-matrix form \( \dot{x} = Ax \) and show that \( A^2 = A \).
Solution:
\[ A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \]
The relation \( A^2 = A \) is a simple verification.

(c) Express the fundamental matrix solution \( \Phi(t) = \exp(At) \) as a linear combination of \( A \) and the identity \( I \).
Solution: The definition is
\[ e^{At} = I + At + \frac{1}{2!} A^2 t^2 + \cdots + \frac{1}{n!} A^n t^n + \cdots. \]
By the identity \( A^2 = A \) we infer \( A^n = A \) for all \( n = 3, 4, \ldots \) so the series becomes
\[ e^{At} = I + A(t + \frac{1}{2!} t^2 + \cdots + \frac{1}{n!} t^n + \cdots), \]
or
\[ e^{At} = I + (e^t - 1) A. \]

6. Consider the differential equation \( w'' - zw = 0 \).

(a) Find the recursion formula determining the coefficients in the power series expansion about the origin.
Solution: With \( w = \sum_0^\infty w_k z^k \) we find
\[ \sum_0^\infty (k + 1)(k + 2)w_{k+2}z^k - \sum_0^\infty w_k z^{k+1} = 0. \]
This gives \( w_2 = 0 \) and for \( k \geq 1 \)

\[
w_{k+2} = \frac{w_{k-1}}{(k+1)(k+2)}.
\]

(b) For initial data \( w(0) = 1, w'(0) = 0 \), find the first three nonvanishing terms of the series.

Solution: \( w(z) = 1 + \frac{1}{6}z^3 + \frac{1}{180}z^6 + \cdots \).

(c) What is the radius of convergence of this series?

Solution: The coefficients are analytic in the entire \( z \)-plane so the radius of convergence is infinite.