1. Problem Set 1.2.1

- 5) This is calculus! \( x(t) = (1/3)t^3 - e^t + 3 \).
- 6) \( x(t) = 100e^{0.05t} \). At \( t = 1 \) this gives 105.13, as compared with 105.12 for monthly compounding.
- 10) If \( \dot{x} = 0 \) then \( x = \gamma/\alpha \). The solution for Problem 7 for arbitrary initial data \( x(0) = k \) is
  \[
  x(t) = (k - \gamma/\alpha)e^{-\alpha t} + \gamma/\alpha,
  \]
  which tends to \( \gamma/\alpha \) as \( t \to \infty \) since \( \alpha > 0 \).
- 12) \( \frac{\partial}{\partial t} f(s + t) = \frac{\partial}{\partial t} f(s + t) = f'(s + t) \), so the partial-differential equation reduces to \( 2f' = \alpha f \). The solution is \( u(s,t) = C \exp((1/2)\alpha(s + t)) \) where \( C \) is an arbitrary constant.
- 14) The solution is \( x(t) = x_0 \exp(\sin t) \); it is periodic.
- 20) Set \( F(u,v) = \int_{t_0}^{t} g(v,s) \, ds \). If \( u = u(t) \) and \( v = v(t) \) and \( f(t) = F(u(t),v(t)) \), the chain rule gives
  \[
  f'(t) = \frac{\partial F}{\partial u} u'(t) + \frac{\partial F}{\partial v} v'(t).
  \]
  Since \( f(t) = F(t,t) \) we put \( u(t) = v(t) = t \), and obtain the stated result.

2. Problem Set 1.3.1

- 1) If \( \dot{x} = k_0 x^{1+\epsilon} \) then
  \[
  x^{-(1+\epsilon)} \frac{dx}{dt} = \frac{d}{dt} \left( -x^{-\epsilon}/\epsilon \right) = k_0,
  \]
  so \( x^{-\epsilon} = C - k_0 t \) where \( C \) is a constant. Evaluating this gives
  \[
  x(t) = \left( x_0^{-\epsilon} - \epsilon k_0 t \right)^{-(1/\epsilon)},
  \]
  which tends to infinity as \( t \) tends to \( t_\ast = x_0^{-\epsilon}/\epsilon k_0 \).
2) The function $G$ of the hint satisfies the condition $G(x_0, y_0) = 0$. Furthermore
\[ \frac{\partial G}{\partial y}(x_0, y_0) = \frac{1}{g(y_0)} \neq 0. \]
Under these conditions the implicit-function theorem guarantees that there exists a unique solution $y = \phi(x)$ of the equation $G(x, y) = 0$ reducing to $y_0$ when $x = x_0$, in a sufficiently small neighborhood of $x_0$.

5) The right-hand side is homogenous of degree zero and the solution, with $y = xv$, is given implicitly by
\[ \int \frac{ds}{\frac{a+bs}{c+ds} - s} = \ln x. \]

10) Since by assumption
\[ \frac{\partial (pM)}{\partial y} = \frac{\partial (pN)}{\partial x} \quad \text{and} \quad \frac{\partial (qM)}{\partial y} = \frac{\partial (qM)}{\partial y}, \]
it follows immediately that the same relation holds with $\alpha p + \beta q$ in place of $p$ or $q$. 