Solutions to Problem Set 5

Answers to Selected Problems in Chapter 4

• 2) The corresponding series are those for \( \exp(-z) \), \( \cosh z \), \( \sinh z \) respectively.

• 5) \( q(z) = -2(1 + z)^{-2} = -2 + 4z - 6z^2 + \ldots \) with \( q_l = 2(l + 1)(-1)^{l+1} \).

From the recursion formula (4.9) of the text with \( w_0 = 1, w_1 = 2 \) one finds that \( w_2 = 1 \) and \( w_3 = 0 \). The claim is that then \( w_k = 0 \) for all \( k \geq 3 \).

This can be proved by induction. It’s true for 3 and assume it’s true for \( n = 3, 4, \ldots k + 1 \). The recursion relation then implies that

\[
(k + 1)(k + 2)w_{k+2} + \{q_2 + 2q_1 + q_0\} = 0.
\]

The sum in curly braces vanishes by virtue of the formula for \( q_l \), completing the induction argument. Thus this solution is \( w(z) = 1 + 2z + z^2 \) as in the example.

• 6) The recursion formula found in this manner is

\[
(k + 2)w_{k+2} + 2kw_{k+1} + (k - 2)w_k = 0.
\]

With the choices \( w_0 = 1, w_1 = 2 \) this gives \( w_2 = 1, w_3 = 0 \) and then \( w_4 = 0 \). Once two consecutive coefficients are zero it is clear from the formula that all subsequent coefficients are also zero,

• 7) Writing the equation in the form \((1 + z^2)w'' + w = 0\) leads easily to the recursion formula

\[
w_{l+2} = -\frac{l^2 - l + 1}{l^2 + 3l + 2} w_l.
\]

The radius of convergence is one.

• 8) The nearest singularity to the origin occurs where the coefficient \( 1 + Az + Bz^2 = 0 \), or at \( z = (-A \pm \sqrt{A^2 - 4B})/2B \). If \( A^2 \geq 4B \) the nearer of the two roots is at a distance

\[
r = \left( |A| - \sqrt{A^2 - 4B} \right) / 2|B|
\]

and if \( A^2 < 4B \) the two roots lie at a distance \( r = 1 \) from the origin. Any power-series solution (in powers of \( z \)) has radius of convergence at least \( r \).

• 9) The recursion formula may be written

\[
(k + 2)(k + 1)w_{k+2} = -\{Bk^2 + (D - B)k + E\} w_k.
\]

The solution may be chosen to be an even function of \( z \) by choosing \( w_0 \neq 0 \) but \( w_1 = 0 \), for then all coefficients with odd indices vanish. Similarly one may choose an odd solution. Suppose for definiteness an even solution is chosen and suppose that the coefficients \( B, D, E \) are such that, for some even integer \( K \), \( BK^2 + (D - B)K + E = 0 \). Then \( W_{K+2l} = 0 \) and therefore \( W_{K+2l} = 0 \) for \( l = 1, 2, \ldots \). The solution is therefore a polynomial of degree \( K \).
13) The recursion formula is

\[ u_{k+2} = \frac{k^2 - \lambda}{(k+1)(k+2)}, \quad k = 0, 1, 2, \ldots \]

If \( \lambda = n^2 \) for an integer \( n \), there is a polynomial solution. If \( n \) is even, put (for example) \( u_0 = 1, u_1 = 0 \). Then all coefficients with odd indices vanish, and coefficients with even indices are nonzero up to and including \( u_n \); all subsequent coefficients vanish. This gives a polynomial in \( z^2 \). Similarly, if \( n \) is odd, one finds an polynomial that is odd in \( z \).

Answers to supplementary problems

1. Let \{\( u_1(z) \)\}_{i=1}^{n} be analytic in a domain \( D \) of the complex plane and suppose their Wronskian determinant vanishes identically there. Show that they are linearly dependent on \( D \) (you may assume the statement is true for \( n = 2 \) and proceed by induction).

We are given that the determinant

\[
\begin{vmatrix}
  u_1 & u_2 & \cdots & u_n \\
  u'_1 & u'_2 & \cdots & u'_n \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{(n-1)}^{(n-1)} & u_{(n-1)}^{(n-1)} & \cdots & u_{(n-1)}^{(n-1)} \\
\end{vmatrix} = 0
\]  

throughout a domain \( D \) of the complex plane. Assume that it is known for \( n - 1 \) in place of \( n \) that the vanishing of this determinant implies that the \( n - 1 \) functions \( u_1, \ldots, u_{n-1} \) are linearly dependent (this was proved in class for two functions). In the present case, if the upper-left-hand \((n-1) \times (n-1)\) determinant above were identically zero in \( D \), then the \( n - 1 \) functions \( u_1 \ldots, u_{n-1} \) would be linearly dependent, by the induction hypothesis, and the conclusion would follow trivially that the \( n \) functions \( u_1, \ldots, u_n \) are linearly dependent. We may therefore assume that this upper left hand determinant (call it \( p_0(z) \)) does not vanish at some point \( z_0 \) in \( D \) and therefore in some domain \( d_0 \subset D_0 \) containing \( z_0 \). The equation (1) can be written

\[ p_0(z)u_n^{(n-1)} + p_1(z)u_{n-2}^{(n-2)} + \cdots + p_{n-1}u_n = 0 \]

on expanding by the last column. Since \( p_0 \) does not vanish on \( d_0 \) this is an equation with analytic coefficients there. Furthermore, the functions \( u_1, \ldots, u_{n-1} \) are solutions (since the equation above is then the expansion of the determinant with two equal columns). Since by assumption they are linearly independent there, we must have

\[ u_n = c_1 u_1 + \cdots + c_{n-1} u_{n-1} \]

on \( d_0 \), and therefore on \( D \) (since an analytic function vanishing on \( d_0 \) must vanish on all of \( D \)).
2. Consider a system \( w' = A(z)w \) where \( A \) is analytic in the disk centered at the origin with radius \( r \), but has a singularity at \( z_0 \) with \( |z_0| = r \) where at least one entry \( a_{ij}(z) \) is unbounded. Show that if a fundamental matrix solution \( \Phi(z) \) remains analytic in a neighborhood of \( z_0 \), its determinant must vanish at \( z_0 \).

Suppose not. Then the equation \( \Phi' = A(z)\Phi \) may be rewritten \( \Phi^{-1}\Phi' = A(z) \), where the left-hand side remains bounded in a neighborhood of \( z_0 \), contradicting the assumption regarding \( A \).

3. Consider the nonlinear, initial-value problem

\[ w' = 1 - 2w + w^2, \quad w(0) = 0. \]

Solve it explicitly to determine the radius of convergence of a power-series representation of the solution.

Solution is \( w(z) = z/(1 + z) \), with radius of convergence 1.