Solutions for Problem Set 6

Answers for problems from Chapter 5 of the text

1. PS 5.1.1

1) Since $\mu_1$ is a double root, we must have the relations $\alpha = 1 - 2\mu_1$ and $\beta = \mu_1^2$. With substitution $w(z) = z^{\mu_1} \ln z$ in the expression $Lw = z^2 w'' + \alpha x w' + \beta w$ you find

$$Lw = z^{\mu_1} \left\{ \left(\mu_1^2 + (\alpha - 1)\mu_1 + \beta \right) \ln z + 2\mu_1 - 1 + \alpha \right\} = 0.$$  

3) The equation becomes the constant-coefficient equation $u'' + (\alpha - 1)u' + \beta u = 0$ where $u(t) = w(z)$. Reading off the solutions as $\exp(\mu_1 t), \exp(\mu_2 t)$ or as $\exp(\mu_1 t), t \exp(\mu_1 t)$ provides the solutions of the theorem.

4) The indicial equation is $\mu^2 - 2\mu + 2 = 0$ with roots $\mu = 1 \pm i$. The solutions are therefore

$$z^{1+i} = ze^{\pm i \ln z}.$$  

For real values of $z$ these have the real and imaginary parts

$$u_1(x) = x \cos(\ln x) \quad \text{and} \quad u_2(x) = x \sin(\ln x).$$

2. PS 5.2.1

5) The indicial equation is $\mu^2 - n^2 = 0$, so $\mu = \pm n$. For $n \geq 0$ an integer, the solution is

$$J_n (z) = \sum_{r=0}^{\infty} \frac{(-1)^r z^{n+2r}}{2^{n+2r} r!(n+r)!}.$$  

The indices do differ by an integer.

6) The singularities of Legendre’s equation occur at $\pm 1$. to check the one at (say) $z = 1$, introduce the new variable $\zeta$ such that $z = 1 + \zeta$. With $w(z) = u(\zeta)$ one finds for Legendre’s equation

$$\zeta^2 u'' + P(\zeta) \zeta u' + Q(\zeta) u = 0$$

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where \( P(\zeta) = \frac{2(1 + \zeta)}{(2 + \zeta)} \) and \( Q(\zeta) = -\lambda \zeta/(2 + \zeta) \). This is a regular singular point since \( P \) and \( Q \) are holomorphic in a neighborhood of \( \zeta = 0 \). Since \( P_0 = 1 \) and \( Q_0 = 0 \), the indicial equation is \( \mu^2 = 0 \), so the index is \( \mu = 0 \) and has multiplicity two. The same result is obtained at the other singular point.

- **8)** The singularities are at \( t = -a_1^2, -a_2^2, -a_3^2 \). Consider the singularity at \( t = -a_1^2 \) and put \( z = t + a_1^2 \). The equation becomes

\[
\zeta^2 v'' + \zeta P(\zeta)v' + Q(\zeta)v = 0
\]

where
\[
P(\zeta) = \frac{1}{2} + \frac{1}{a_2^2 - a_1^2 + z} + \frac{1}{a_3^2 - a_1^2 + z}
\]
and
\[
Q(\zeta) = \frac{A \zeta + B - a_1^2 A}{(a_2^2 - a_1^2 + z)(a_3^2 - a_1^2 + z)}.
\]
This gives \( P_0 = 1/2 \) and \( Q_0 = 0 \) so the indicial equation is
\[
\mu^2 - (1/2)\mu = 0
\]
with roots \( \mu = 0, 1/2 \).

3. **PS 5.3.1**

- **2)** Using the given information one can express \( w''' \) in terms of \( w_1, w_2, w_3 \) and \( w_3' \) to get the indicated equation with

\[
A(z) = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-a_3 & (a_1 - a_2 - 2) & (3 - a_1)
\end{pmatrix}.
\]

- **3)** The equation has singularities only at \( z = \pm 1 \). To study that at \( z = +1 \) put \( \zeta = z - 1 \) and \( w(z) = u(\zeta) \). The equation becomes (after multiplying through by \( \zeta/(2 + \zeta) \))

\[
\zeta^2 u'' + \zeta P(\zeta)u' + Q(\zeta)u = 0,
\]
where \( P(\zeta) = (1 + \zeta)/(2 + \zeta) \) and \( Q(\zeta) = -\lambda \zeta/(2 + \zeta) \). Therefore \( P_0 = 1/2, Q_0 = 0 \) and the indicial equation is \( \mu(\mu - 1/2) = 0 \). The indices relative to this point \( (z = +1) \) are therefore \( \mu = 0 \) and \( \mu = 1/2 \).

A similar analysis for the point \( z = -1 \) gives the same indices.
4) The equation is

\[(1 - z)z^2w'' + (z - 4)zw' + 6w = 0 \quad (1)\]

- (a) The equation may be written

\[z^2w'' + zP(z)w' + Q(z)w = 0\]

by dividing through by \(1 - z\). The functions \(P = (z - 4)/(1 - z)\)
and \(Q(z) = 6/(1 - z)\) are analytic in a neighborhood of the
origin so the equation is regular (or rather “Fuchsian”) at
\(z = 0\).

- (b) The indicial polynomial is \(I(\mu) = \mu^2 - 5\mu + 6 = 0\) so the
indices are \(\mu = 2\) and \(\mu = 3\).

- (c) Instead of using the general expressions for the recursion
relation it is simpler in this case to restore the original form
(1) and derive the recursion relation from this. Under the
assumption \(w = z^\mu \sum a_k z^k\) one finds, for \(k = 0\)

\[I(\mu)a_0 = 0\] so, since \(a_0 \neq 0\), \(I(\mu) = 0\). \quad (2)

For \(k = 1, 2, \ldots\),

\[I(\mu + k)a_k = (k - 1 + \mu)(k - 3 + \mu)a_{k-1} \quad (3)\]

Equation (2) is satisfied for the larger index \(\mu = 3\) and equation (3) determines all further coefficients once \(a_0\) is assigned.

- (d) Suppose we take \(\mu = 2\) above. The equation (2) is satis-
fied with \(a_0 \neq 0\), but we might encounter a problem for the
determination on the next coefficient \(a_1\) since for \(\mu = 2\) we
have \(I(\mu + 1) = I(3) = 0\). By equation (3) above we have

\[I(3)a_1 = (2)(0)a_0,\]

so the right-hand side vanishes and this is satisfied with an
arbitrary choice of \(a_1\), i.e., we do \textit{not} run into the anticipated
problem. Making such a choice and continuing provides a
formal solution which, by the convergence theorems of §5.3.1,
is an actual solution. These two solutions are linearly inde-
pendent since their leading-order terms \(z^3\) and \(z^2\) imply a
nonvanishing Wronskian in a punctured neighborhood of the
origin.
8) With $\zeta = 1/z$ and $u(\zeta) = w(z)$ one finds

$$\zeta^2 u'' + 2\zeta u' + u = 0.$$ 

There is a regular (Fuchsian) singularity at $\infty$. The indices are $\mu = (-1 \pm i\sqrt{3})/2$. 