1. Solve explicitly the following first-order initial-value problems:

   (a) \( y' - xy = 2x, \quad y(0) = 1 \).

   (b) \( y' = (x + y)^2, \quad y(0) = 0 \).

2. The number \( x_0 \) is a double zero of the function \( u(x) \) if \( u(x_0) = 0 \) and \( u'(x_0) = 0 \). Suppose \( u \) is a solution, not identically zero, of the differential equation

\[
 u'' + p(x)u' + q(x)u = 0
\]

on the interval \([a, b]\) and assume that \( p \) and \( q \) are continuous functions on this interval. Can \( u \) have a double zero at some point of this interval? If not, why not; if so, construct an example.

3. Find the influence function providing a particular integral of the differential equation

\[
 u'' + k^2 u = r(x),
\]

where \( k \) is a nonzero constant and \( r \) is an arbitrary continuous function. Express the most general solution of this equation.

4. The motion of a mass on a spring near its rest state \((y = 0)\) is described by the equation

\[
 Ly \equiv \ddot{y} + \nu \dot{y} + y = \cos(\omega t),
\]

where \( \nu \) is a (positive) coefficient of friction, and \( \omega \neq 1 \). Find a particular integral. What modification would have to be made if \( \nu = 0 \) and \( \omega = 1 \)?

5. For the equation

\[
 (1 + Ax + Bx^2)u'' + (C + Dx)u' + Eu = 0
\]

where \( A, B, C, D, E \) are real constants,

   (a) what is the minimum radius of convergence of power series solutions \( \sum a_kx^k \) ?

   (b) assuming \( A = C = 0, B = 1 \) find a condition on the remaining constants \((D, E)\) for this equation to possess a polynomial solution.

6. For the equation

\[
 z^2(1 - z)^2w'' - z(z - 1)(2z - 1)w' + (2z^2 - 2z + 1)w = 0,
\]

(a) Locate the singular points in the finite complex plane and determine which are regular.

(b) Obtain the indicial equation relative to each regular singular point.
(c) Indicate the qualitative behavior of solutions near each singular point (i.e., are all solutions bounded, all unbounded, some bounded and some unbounded?).

7. Bessel’s equation of order \( n = 1/2 \) is
\[
 z^2 w'' + zw' + \left( z^2 - \frac{1}{4} \right) w = 0.
\]
Find a basis of solutions by the method of Frobenius. Does the second solution have a logarithmic term? Express the solutions in terms of elementary functions.

8. Consider the nonlinear analytic initial-value problem
\[
 w'' = 2w^3, \quad w(0) = 1, \quad w'(0) = 1.
\]
(a) Assume that it has a power-series solution valid near \( z = 0 \) and find it through terms of order \( z^4 \).
(b) Find an explicit, exact solution and thereby determine the radius of convergence of the power series of part a.

9. For the equation \( zw'' + w = 0 \)
(a) Locate the singular points of this equation in the finite complex plain and, for regular singular points, find the indicial equations and the indices.
(b) Find recursion formulas for a linearly independent pair of solutions valid near \( z = 0 \).

10. Consider the one-dimensional initial-value problem
\[
 \frac{dy}{dx} = f(x, y), \quad y(a) = \alpha
\]
where the function \( f \) is \( C^1 \) and bounded on the vertical strip
\[
 S = \{x, y : a \leq x \leq b, \ y \text{ unrestricted}\}.
\]
Show that a unique solution exists on all of \([a, b]\).

11. Consider the regular Sturm-Liouville problem on the interval \([a, b]\):
\[
 \frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + (\lambda p(x) - q(x)) u = 0, \quad (1)
\]
\[
 \alpha u(a) + \alpha' u'(a) = 0, \quad \beta u(b) + \beta' u'(b) = 0, \quad (2)
\]
where \( \alpha, \alpha' \) are not both zero, and likewise for \( \beta, \beta' \).
Suppose \( \{\lambda_n\}_{n=0}^{\infty} \) is the sequence of eigenvalues and \( \{u_n\}_{n=0}^{\infty} \) the corresponding sequence of eigenfunctions.
(a) Derive the orthogonality relation
\[ \int_a^b \rho(x) u_n(x) u_m(x) \, dx = 0 \quad n \neq m. \]

(b) If for a given function \( f(x) \) defined on \([a, b]\) it is true that \( f(x) = \sum_{k=0}^{\infty} c_k u_k(x) \) for certain constant coefficients \( c_k \), find an expression for \( c_k \) (you may take for granted issues of convergence).

12. Find the eigenvalues and eigenfunctions of the following boundary-value problem:
\[ u'' + x^{-2} \lambda u = 0, \quad 1 < x < 2; \quad u(1) = 0, \; u(2) = 0. \]

13. Consider Airy’s equation \( u'' + xu = 0 \), on the intervals \( I_1 = (-\infty, 0) \) and \( I_2 = (0, +\infty) \).

   (a) On which interval is the equation non-oscillatory (this means that any solution has at most one zero on the interval)?

   (b) How many zeros do solutions of Airy’s equation have on the other (oscillatory) interval?

   (c) If \( \xi \) and \( \eta \) are consecutive zeros on the oscillatory interval, find an estimate for the distance \( |\eta - \xi| \) between them.

14. In the Sturm-Liouville problem (1), (2), suppose \( \alpha' = \beta = 0 \). Under what conditions on the coefficients is it true that all eigenvalues are positive?