Supplementary Problems for Chapter 3

1. Let $A$ be the $n \times n$ companion matrix as in equation (3.65) of the text. Suppose $\lambda$ is an eigenvalue of multiplicity two and let $\xi$ be an eigenvector: $A\xi = \lambda \xi$. You may regard as established that there is a second, linearly independent, vector $\eta$ such that $A\eta = \lambda \eta + \xi$. Show that the time-dependent function $x(t) = \exp(\lambda t)\xi$ is a solution of the equation $\dot{x} = Ax$. Find a second, linearly independent solution of this equation as a linear combination of $\xi$ and $\eta$ with coefficients depending on $t$.

2. Consider the system of 3 equations

$$\frac{dy}{dt} = Ay$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}.$$ 

Find the fundamental matrix solution reducing to the identity when $t = 0$. Express it as a linear combination of the identity $I$ and of $A$ itself, with coefficients that depend on $t$.

3. Let $A$ be the matrix of problem 6 in Problem Set 3.6.1. Suppose that $\lambda < 0$ and consider the equation $\dot{x} = Ax$.

(a) Show that all solutions decay to zero as $t \to \infty$.

(b) Define $\|y\| = \sqrt{y_1^2 + y_2^2}$. Show that there exist initial data $y(0)$ such that, if $|\lambda|$ is small enough, the maximum value attained by the ratio $\|y(t)\|/\|y(0)\|$ can be made as large as we please.