1. Find the solutions of the initial-value problems on an interval including the origin:

(a) \( y' = xy - x \), \( y(0) = 1 \).

SOLUTION: \( y(x) = 1 \) is a solution, by inspection.

(b) \( y' = -xy^{-1} \), \( y(0) = 1 \).

SOLUTION: \( yy' = -x \) so \( y^2 = -x^2 + C \). The initial data imply that \( C = 1 \) so \( y = \sqrt{1 - x^2} \).

2. Consider the set of real-valued functions

\[ u_1(x), u_2(x), \ldots, u_n(x), \quad a \leq x \leq b. \quad (1) \]

(a) What does it mean for these to be linearly independent on \([a, b]\)?

SOLUTION: IN THE RELATION OF LINEAR DEPENDENCE

\[ c_1u_1(x) + \cdots + c_nu_n(x) = 0 \quad (2) \]

FOR ALL \( x \) IN \([a, b]\), WITH CONSTANT COEFFICIENTS \( c_1, c_2, \ldots, c_n \), THESE COEFFICIENTS ARE ALL ZERO.

(b) Suppose this set of functions is orthogonal:

\[ \int_a^b u_i(x)u_j(x) \, dx = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (3) \]

Show that they are linearly independent.

SOLUTION: IN EQUATION (2) MULTIPLY THROUGH BY \( u_k(x) \) AND INTEGRATE: THIS GIVES \( c_k = 0 \). THIS CAN BE DONE FOR EACH \( k = 1, 2, \ldots, n \) SO EACH COEFFICIENT VANISHES.

3. Let \( Lu = u'' - x^{-1}u' \) on the interval \( x \geq 1 \).

(a) Find a basis of solutions for the equation \( Lu = 0 \).

SOLUTION: THIS IS A FIRST-ORDER EQUATION IN \( v = u' \) : \( v' = x^{-1}v \) WITH A SOLUTION \( v = x \). THEREFORE THE GENERAL SOLUTION IS \( u = ax^2 + b \) FOR CONSTANTS \( a, b \) AND A BASIS IS \( u_1 = 1, u_2 = x^2 \).
(b) Find the influence function $K(x, s)$ for a solution of the inhomogeneous equation $Lu = r(x)$.

SOLUTION: $K(x, s) = c_1(s)u_1(x) + c_2(s)u_2(x) = c_1(s) + c_2(s)x^2$

WITH INITIAL DATA $K(s, s) = 0, K_2(s, s) = 1$. SOLVING THESE GIVES

$$K(x, s) = \left(\frac{1}{2}\right) \left(-s + \frac{x^2}{s}\right).$$

4. Let $u$ and $v$ be a basis of solutions of the equation $u'' + p(x)u' + q(x)u = 0$ on the interval $[a, b]$, where $p, q$ are continuous. Show that the zeros of $u$ and $v$ separate each other, i.e., between two consecutive zeros of $u$ there is one (and only one) zero of $v$.

SOLUTION: THE WRONSKIAN $w = uv' - u'v$ DOES NOT VANISH; SUPPOSE IT'S POSITIVE. LET $x_1$ AND $x_2$ BE CONSECUTIVE ZEROS OF $u$, SO THAT $u$ DOES NOT VANISH ON $(x_1, x_2)$: SUPPOSE IT'S POSITIVE THERE. CONSIDER $w(x_1) = -u'(x_1)v(x_1)$. $u'(x_1) > 0$ IN ORDER FOR $u$ TO BE POSITIVE ON THE INTERVAL, AND THAT IMPLIES THAT $v(x_1)$ IS NEGATIVE. AT THE OTHER ENDPOINT $x_2$, WHERE $u'$ MUST BE NEGATIVE, IT NOW FOLLOWS THAT $v(x_2) < 0$. SINCE $v$ HAS DIFFERENT SIGNS AT THE ENDPOINTS IT VANISHES AT LEAST ONCE IN BETWEEN. IT CANNOT VANISH MORE THAN ONCE FOR IF IT DID WE COULD APPLY THIS SAME ARGUMENT REVERSING THE ROLES OF $u$ AND $v$ AND INFER THAT $u$ VANISHES AT LEAST ONCE ON $(x_1, x_2)$, CONTRARY TO HYPOTHESIS.

5. Find any solution of the equation $u'' + 2u' + u = x^2$.

SOLUTION: THERE MUST BE A PARTICULAR INTEGRAL OF THE FORM

$$u = Ax^2 + Bx + C$$

BY THE METHOD OF UNDETERMINED COEFFICIENTS. SUBSTITUTING THIS GIVES

$$A = 1, B = -4, C = 6 \text{ or } u_P = x^2 - 4x + 6.$$
THE FULL SOLUTION, INCLUDING A LINEAR COMBINATION
OF SOLUTIONS OF THE HOMOGENEOUS EQUATION IS

\[ u(x) = u_P(x) + c_1 e^{-x} + c_1 x e^{-x}. \]

6. Consider the system of equations

\[ \begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_2.
\end{align*} \quad (4) \]

(a) Find the most general solution of this system.

**SOLUTION:** THE SECOND GIVES \( x_2 = c_2 e^t \) SO THE SOLUTION IS
\[ x_1 = c_1 + c_2 e^t, \quad x_2 = c_2 e^t. \]

(b) Write the matrix \( A \) expressing this system in the vector-matrix
form \( \dot{x} = Ax \) and show that \( A^2 = A \).

**SOLUTION:** \( A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \), AND IT IS EASY TO CHECK
THAT \( A^2 = A \).

(c) Express the fundamental matrix solution \( \Phi(t) = \exp(At) \) as a
linear combination of \( A \) and the identity \( I \).

**SOLUTION:** SINCE \( A^2 = A \) IT FOLLOWS THAT \( A^k = A \) AND
THE POWER SERIES FOR \( e^{At} \) BECOMES
\[ I + tA + \frac{t^2}{2!} A + \cdots + \frac{t^k}{k!} A + \cdots = I + (e^t - 1)A. \]