Answers to Selected Problems in Chapters 1 and 2

• Problem Set 1.4.1

  5) The first part of the question is a verification. For the second, note that for the function to satisfy a Lipschitz condition on \([0, a)\) it would be necessary that

  \[ |f(x) - f(0)| = \frac{x^2}{3} \leq Lx \quad \text{for some constant} \quad L, \]

  or that \( L \geq x^{-1/3} \) on this interval. This is not possible.

• Problem Set 2.1.1

  4) For \( x < 0 \) \( v'' = -6x \) and for \( x > 0 \) \( v'' = 6x \) so the limit of \( v'' \) is zero. It is a simple calculation that \( v'(0) = v''(0) = 0 \), and this verifies the continuity of \( v'' \). The Wronskian is zero at each point of the interval but the functions are linearly independent.

  7) Unless all the coefficients \( c_1, \ldots, c_k \) vanish the expression could vanish only at a finite number of points whereas it is required to vanish on an interval.

  10) 
      * a) \( u_1(x) = 1, u_2(x) = x \)
      * b) \( u_1(x) = 1, u_2(x) = \exp(-2x) \)
      * c) \( u_1(x) = 1, u_2(x) = \int_0^x \exp(-s^2/2) \, ds \)

  13) It is a verification that the given functions are solutions. Their Wronskian is \(-2/x^4\), which is nowhere zero and in particular does not vanish on any interval excluding the origin.

  14) That each is a solution is a verification. Their Wronskian is

  \[ (\sin x - x \cos x)x \cos x - (\cos x + x \sin x)x \sin x = -x^2, \]

  nonvanishing on any interval excluding the origin.

  20) Suppose to the contrary \( u \) has more than one zero and let \( x_1 \) and \( x_2 \) be consecutive zeros. We must have \( u'(x_1) \neq 0 \) since otherwise \( u \) would vanish identically; assume for definiteness that \( u'(x_1) > 0 \). Then, since \( x_1 \) and \( x_2 \) are consecutive zeros, \( u > 0 \) on
Since it vanishes at $x_2$, it must have a positive maximum at (say) $x_*$ in $(x_1, x_2)$. Then $u'(x_*) = 0$ whereas $u''(x_*) \leq 0$. But by the differential equation
\[ u''(x_*) = -q(x_*)u(x_*) > 0, \]
a contradiction.

21) The pair of equations
\[
\begin{align*}
  u'p(x) + uq(x) &= -u'', \\
  v'p(x) + vq(x) &= -v''
\end{align*}
\]
have unique solutions for $p$ and $q$ provided $u'v - uv' = W$ does not vanish; this provides the coefficients $p, q$.

- Problem Set 2.1.2
  1) A particular integral is $U(x) = x$ and the most general solution is $u = x + c_1 \cos x + c_2 \sin x$. The solution with the given initial data is $u = x - \sin x$.
  3) The equation may be written
\[ u'' + \frac{p'}{p}u' + \frac{q}{p} = \frac{r}{p}, \]
so a particular integral is
\[ U(x) = \int_a^x \frac{u_1(s)u_2(x) - u_1(x)u_2(s)}{W(s)} \frac{r(s)}{p(s)} ds. \]
Since one finds that $W(s) = c/p(s)$ where $c$ is a constant, this becomes
\[ U(x) = \int_a^x \frac{u_1(s)u_2(x) - u_1(x)u_2(s)}{c} r(s) ds. \]

- Problem Set 2.3.1
  1) The Wronskian is $W = 2$ and is constant because the coefficient $a_1(x)$ vanishes.