Solutions for Problem Set 5

- PS 6.2.1

- 3) The equivalent integral equation is

\[ y(x) = 1 + \int_0^x y(s)^2 \, ds, \]

so \( y_0(x) = 1 \) and

\[ y_1(x) = 1 + \int_0^x 1 \, ds = 1 + x. \]

Therefore

\[ y_2(x) = 1 + \int_0^x (1 + s)^2 \, ds = 1 + x + x^2 + \frac{1}{3}x^3. \]

- 7) This separable, first-order equation is easily solved:

\[ u(x) = \left\{ \frac{3}{2} + x \right\}(\frac{5}{2} - x) \]

giving \((-3/2, +5/2)\) as the maximal interval.

- 8) Differentiating each side of the \( u' \) equation and using the \( v' \) equation gives an equation relating \( u'' \) to \( u', u, \) and \( v; \) a second use of the original \( u' \) equation then eliminates \( v, \) giving

\[ u'' + p(x)u' + q(x)u = 0, \]

where

\[ p(x) = -(a + d + b'/b), \quad q(x) = -(a' + bc - ad - ab'/b). \]

In addition to the differentiability requirements, it is necessary that \( b(x) \) not vanish on \( I. \)

- 9) For \( n = 2 \) we have

\[ \|u\|^2 = u_1^2 + u_2^2 \leq u_1^2 + 2|u_1||u_2| + u_2^2 = (|u_1| + |u_2|)^2 \]

so \( \|u\| \leq |u|. \) Since \( 2|u_1||u_2| \leq u_1^2 + u_2^2 \) we have also

\[ (|u_1| + |u_2|)^2 \leq 2 \left( u_1^2 + u_2^2 \right) = 2\|u\|^2, \]

\[ |u| \leq \sqrt{2}\|u\|. \] The numbers \( a, b, c, d \) of the problem can be chosen to be \( 1, \sqrt{2}, 1/\sqrt{2}, 1. \)
13) The domain may regarded as unrestricted, so either a solution becomes infinite at the endpoints $a, b$ of a maximal interval, or these numbers may be replaced by $-\infty, +\infty$. Consider the quantity $E = Aw_1^2 + Bw_2^2 + Cw_3^2$ for as yet unspecified constants $A, B, C$. Then

$$\frac{dE}{dx} = 2(Aw_1w'_1 + Bw_2w'_2 + Cw_3w'_3) = 2w_1w_2w_3(A - B - \mu C).$$

Choosing, e.g., $A = 1 + \mu, B = C = 1$, shows that positive quantity

$$E = (1 + \mu)w_1^2 + w_2^2 + w_3^2$$

is constant on solutions. If the solution were unbounded this quantity would likewise be unbounded, but is constant. Hence solutions exist for all real $x$.

- **PS 6.4.1**

  1) The solution is given in the text and its continuity as a function of $u_0$ is obvious.

  2) $\phi(x - x_0, U_0)$ satisfies the differential equation $U' = F(U)$ if $\phi(x, U_0)$ does and satisfies the initial data, so the result follows from the uniqueness theorem.

  5) Using the formula for $u$ given in the problem and differentiating with respect to $u_0$, we find

$$v = \frac{\partial u}{\partial u_0} = (1 - u_0x^2)^{-2}.$$

On the other hand, the variational equation for $v = \partial u/\partial u_0$ is found to be

$$v' = 4xuv = 4xu_0 \frac{x}{1 - u_0x^2}v, \quad v(0) = 1.$$

This is a first-order linear equation and easily integrable: the result is the same as the formula above.

8) These are easily solved if we write $\dot{x}_1 = -\delta x_1 + x_2, \dot{x}_2 = -\mu x_2$.

  * (a) For $\mu = \delta$, $x_1 = te^{-\delta t}, x_2 = e^{-\delta t}$.  
* (b) For $\mu \neq \delta$, $x_1 = \left(e^{-\mu t} - e^{-\delta t}\right) / (\delta - \mu)$ and $x_2 = e^{-\mu t}$.
* (c) For fixed $t$, the expression for $x_1$ above is the difference quotient for the derivative of $e^{-\delta t}$ with respect to $\delta$, which gives $te^{-\delta t}$, as advertised.

9) With $\mu = 0$ the original system becomes

$$
\dot{w}_1 = w_2 w_3, \quad \dot{w}_2 = -w_1 w_3, \quad \dot{w}_3 = 0.
$$

Thus $w_3 = 1$ and the first two equations give $w_1 = w_2 = 0$. The variational equations for $v = \partial w / \partial \mu$ become

$$
\dot{v}_1 = w_2 v_3 + w_3 v_2, \quad \dot{v}_2 = -w_1 v_3 - w_3 v_1, \quad \dot{v}_3 = -w_1 w_2 - \mu w_1 v_2 - \mu w_2 v_1.
$$

Using the values found for $w$ now gives the equations

$$
\dot{v}_1 = v_2, \quad \dot{v}_2 = -v_1, \quad \dot{v}_3 = 0.
$$

The solution is $v_1 = c_1 \cos t + c_2 \sin t, v_2 = -c_1 \sin t + c_2 \cos t, v_3 = c_3$ where $c_1, c_2, c_3$ are arbitrary constants.