Supplementary Problems for Chapter 10

1. Consider the equation
\[ u'' = \lambda u, \quad 0 < x < \pi. \]

Find solutions \( u_1, u_2 \) satisfying the following initial conditions:
\[ u_1(0) = 0, \quad u_1'(0) = 1; \quad u_2(\pi) = 0, \quad u_2'(\pi) = 1. \]

You may allow \( \lambda \), as well as the values of \( u_1 \) and \( u_2 \), to be complex.
Calculate the Wronskian of these solutions and determine when \( u_1 \) and \( u_2 \) are linearly dependent.

2. Consider the equation
\[ f(x)u'' + g(x)u' + h(x)u = \phi(x), \quad a < x < b \quad (1) \]
where \( f(x) > 0 \) on \((a, b)\). Find an 'integrating factor' \( V(x) \) putting equation (1) into the form
\[ (p(x)u')' + V(x)h(x)u = V(x)\phi(x), \quad a < x < b. \quad (2) \]

Make whatever smoothness assumptions you need about \( f, g, h, \phi \).

3. For equation (2) introduce a change of variables \( x = X(\xi) \) and put \( u(x) = v(\xi) \) so that this equation takes the form
\[ \frac{d^2v}{d\xi^2} + Q(\xi)v = \psi(\xi). \]
Again make whatever smoothness assumptions you need. Express \( Q, \psi \) in terms of the functions appearing in equation (2).