Supplementary Problems for Problem Set 6

For the first two problems recall Liouville’s formula for the evolution of volumes in phase space under the action of the system $\dot{x} = f(x,t)$:

$$V(t) = \int_{V_0} \exp \left( \int_{t_0}^t \text{div} f(\phi(s,x_0), s) \, ds \right) \, dx_0,$$

where $\phi(t,x_0)$ represents the solution of the system with initial data $x_0$ at time $t_0$.

1. For the system

$$\dot{x} = x, \quad \dot{y} = -y$$

consider a cloud of initial data (at $t = 0$) in the rectangle with corners

$$(x_0, y_0), (x_0 + h, y_0), (x_0 + h, y_0 + k), (x_0, y_0 + k)$$

and describe the region into which it evolves at time $t > 0$ under the action of the differential equations. What can you infer regarding the evolution of areas in phase space? Explain how you could arrive at your conclusion using Liouville’s formula.

2. For the system (Euler’s equations of rigid-body dynamics)

$$\dot{\omega}_1 = \omega_2 \omega_3, \quad \dot{\omega}_2 = -\omega_1 \omega_3, \quad \dot{\omega}_3 = -\mu \omega_1 \omega_1$$

how do volumes evolve in phase space?

3. Solve the one-dimensional system

$$\dot{x} = -x^2$$

explicitly for $x = \phi(t,x_0)$ and verify condition (iii) of (7.6).

4. Suppose you are given a smooth function $\phi(t,y)$ satisfying the conditions (7.6). Show that there is a function $f(y)$ of $y$ only – and find it! – such that $\phi(t,x)$ is the solution of the initial-value problem

$$\dot{y} = f(y), \quad y(0) = x.$$