

Natural quasirandomness properties

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Abstract

The theory of quasirandomness has greatly expanded from its inaugural graph theoretical setting to several different combinatorial objects such as hypergraphs, tournaments, permutations, etc. However, these quasirandomness variants have been done in an ad-hoc case-by-case manner. In this paper, we propose three new hierarchies of quasirandomness properties that can be naturally defined for arbitrary combinatorial objects. Our properties are also “natural” in more formal sense: they are preserved by local combinatorial constructions (encoded by open interpretations). We show that our quasirandomness properties have several different but equivalent characterizations that are similar to hypergraph quasirandomness properties. We also prove several implications and separations comparing them to each other and to what has been known for hypergraphs.

The main notion explored by our statements and proofs is that of unique coupleability: two limit objects are uniquely coupleable if there is a unique limit object in the combined theory that is an alignment (i.e., a coupling) of these two objects.

1 Introduction

The theory of graph quasirandomness introduced by Thomason [Tho87] and Chung–Graham–Wilson [CGW89] studies deterministic graphs that look

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random. The main discovery of this theory is that several properties that hold asymptotically almost surely for the sequence of Erdős–Rényi random graphs $(\mathbf{G}_{n,p})_{n \in \mathbb{N}}$ are equivalent when rephrased as properties of a deterministic graph sequence $(G_n)_{n \in \mathbb{N}}$. Since then, the theory of quasirandomness has expanded not only within graph theory [CG92, SS97, SS03, Sha08, Yus10, Jan11, CFS18] but also towards studying quasirandomness for other combinatorial objects such as tournaments [CG91, KS13, CR17], permutations [Coo04, Coo05, KP13, CKN⁺20] and hypergraphs [CG90, Chu90, KRS02, KNRS10, DR11, LM15b, LM17, Tow17, AHCH⁺18].

The theory of quasirandomness was one of the motivations and driving forces behind the theory of dense limits of combinatorial objects (we refer the reader to [Lov12] for the case of graphs and to [Aus08, AC14, CR20] for the general case). The starting point of the latter theory is that if $(N_n)_{n \in \mathbb{N}}$ is a sequence of combinatorial objects such that for every fixed combinatorial object M , the normalized number of (unlabeled induced) copies $p(M, N_n)$ of M in N_n converges to some limit $\phi(M)$, then the sequence $(N_n)_{n \in \mathbb{N}}$ can alternatively be represented as a limit object that captures all these limit values. But as the theory of graph (and other) limits has been maturing, and in particular after the uniqueness theorem was proved in [BCL10] (see [Lov12, Theorem 13.10] for graphs and [CR20, Theorem 3.9] for the general case), it has turned out that in a sense this theory transcends counting. Namely, limit objects can be described, up to an appropriate notion of isomorphism (or, as Lovász dubbed it, *cryptomorphism*), using different languages and quite different kinds of mathematics and statistics ([Lov12, Theorem 11.52] and [CR20, Theorem 6.3]) and only one of those descriptions is based on sampling statistics $p(M, -)$ per se [Raz07]. Arguably, it is this versatility that is largely responsible for the wide spread of graph limits and their connections to many other areas.

The situation with quasirandomness remains somewhat different, and we are aware only of a few attempts to study it intrinsically, that is, based on principles other than counting. One of the equivalent properties in the seminal paper [CGW89] (P_3) was of spectral nature, namely it requested the second largest eigenvalue of G_n to be $o(|G_n|)$. This spectral theme was further continued for (linear) quasirandom hypergraphs in [LM15a, LM17].

Even though most other quasirandomness properties in the literature are stated in terms of counting, it is still possible to extract from them something intrinsic. For example, the property P_4 in [CGW89] (see also [SS97,

Theorem 2.4]) implies that quasirandom limits W are the only graphons with the following unique inducibility property: if $(G_n)_{n \in \mathbb{N}}$ converges to W then the sequence of induced graphs $(G_n|_{U_n})_{n \in \mathbb{N}}$ also converges to W as long as $|U_n| \geq \Omega(|G_n|)$. As another example, using graphon language [LS06], we can extract a trivial intrinsic characterization of quasirandom limits in terms of an independence property: a graphon $W: [0, 1]^2 \rightarrow [0, 1]$ is quasirandom if and only if W a.e. does not depend on its variables, that is, it is a.e. constant.

In this paper we attempt to initiate a more systematic study of quasirandom properties that can be reasonably identified as “intrinsic” (for reasons that will become clear very shortly, we will also use in this context the word “natural”), and let us first explain what we roughly mean by this. Our explanation will be deliberately informal and open-ended; instead of trying to give a rigorous definition, we present a set of tests that in our view have to be passed and then describe some concrete properties we will be studying in this paper that pass these tests.

First and foremost, we view this paper as a continuation of [Raz07, CR20], which in particular implies that we require qualifying properties to be formulated in an uniform way for arbitrary universal theories in a finite relational language. For examples of what can be expressed in that language see [CR20, Sct. 2.1 and Sct. 7].

The next two requirements are somewhat derivative of the first.

We require that the property should not refer to densities of concrete models and their explicit values (thus, this is more about the *formulation* of the property than the class of objects defined by it.) The reason is that any such definition is necessarily somewhat arbitrary. For example, there is no such thing as “edge densities” in the theories of tournaments and permutations so their ad hoc analogues had to be found when defining quasirandom objects in those contexts. Of the quasirandom graph properties mentioned above, the description as a constant graphon definitely satisfies this criterion, and so does the inducibility property (the tweak of P_4 in [CGW89]). Spectral properties also pass the test but unfortunately they fail (given our current state of knowledge) the previous universality test.

The next requirement is that we want the property to be preserved under open interpretations, and this is where the word “natural” (like in “natural transformations” – open interpretations do form a category [CR20, Sct. 2.2]) comes in. In plain words, everything that can be syntactically defined in a quasirandom object must display proportionally strong quasirandom

properties. Again, in an implicit form this requirement was exploited in the previous literature both in positive and negative manner. For example, the proofs of the implications $P_{10} \implies P_{11} \implies P_1(s)$ in the seminal paper on quasirandom tournaments [CG91] can be viewed as divided into two parts. First one proves that all “couplings” of a quasirandom graph with a linear ordering are the same and hence completely determined by the random coupling. Then the tournament obtained from the resulting quasirandom ordered graph via the “arc-orientation” interpretation must be quasirandom. This example is paradigmatic for many parts of our paper. As for “negative” use, let us note that most separations in the hierarchy of quasirandom hypergraphs [AHCH⁺18, LM15b, Tow17] can be viewed as coming from the fact that these properties are *not* preserved under open interpretations between the theories of hypergraphs of possibly different arity. We will elaborate on this in Section 8 (see Theorem 3.15).

Our final requirement is more “traditional”, and it is well-rooted in the previous literature. Namely, we require that the property should be satisfied asymptotically almost surely for some “natural” random model of some “natural” theory T . Examples of “natural” random models include, of course, the Erdős–Rényi model and its generalization to hypergraphs, the random tournament, the random permutation, etc.

This list of requirements may appear to be rather restrictive, so let us describe quasirandom properties we are studying; they are essentially far-reaching generalizations of what we already discussed above. Several more remarks are in place before we begin.

1. We have deliberately decided against attempting to state our properties in the language of finite combinatorial objects and their asymptotic behavior – it is probably possible but the result might be rather ugly and disappointing. Instead, we use the language of graphons [LS06], hypergraphons [ES12] and theons [CR20] for the geometric view of our objects and that of flag algebras [Raz07] for a concise algebraic description. We remark that we are not the first authors to make this election, and the advantages of using the continuous setting are illustrated by the fact that such proofs are often more elegant and less technical than their finite world counterparts [Jan11, KP13, Tow17]. This view is more instructive, too: for example, by looking back through the lenses of graphons, we can extract an elegant graphon proof of

quasirandomness of property $P_2(4)$ of [CGW89] based on the Lebesgue Density Theorem from a paper as early as [DF81, Theorem 3.10].

However, for the benefit of more combinatorially-oriented reader we try to inject as much of “finite intuition” as possible in appropriate places.

2. Our properties are not equivalent with those previously studied in the literature even for hypergraphs (see Figure 2). Hence the reader interested only in this case can safely assume that our base theory is $T_{k\text{-Hypergraph}}$ for some $k \geq 3$, and the objects are just hypergraphons. But let us mention that more complicated objects like colorings, orderings, couplings, etc. will pop up in the statements and the proofs anyway.
3. Finally, the description below is loose and sweeps under the rug some important technicalities. Proper definitions are deferred to Section 2.2.

Independence[ℓ] If we want to realize the quasirandom (that is, constant) graphon of density p as a 2-hypergraphon $\mathcal{G} \subseteq [0, 1]^3$, one way of doing it is by

$$\mathcal{G} \stackrel{\text{def}}{=} \{ (x_{\{1\}}, x_{\{2\}}, x_{\{1,2\}}) \mid x_{\{1,2\}} \leq p \}. \quad (1)$$

This 2-hypergraphon has one peculiar property: it does not depend on first-order coordinates $x_{\{1\}}, x_{\{2\}}$, and this property is perfectly generalizable. Namely, we call a combinatorial object ϕ ℓ -independent if it has a representation similar to (1) that does not depend on the coordinates x_A with $|A| \leq \ell$. This is the strongest in the hierarchy of our properties, and it relatively easily implies all the others, with the same value of the parameter ℓ . Let us also remark that if the object is given in an implicit form, say as a positive homomorphism $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ from the flag algebra, then **Independence**[ℓ] only talks about the *existence* of the required geometric realization or, equivalently, about the *possibility* of straightening up any geometric realization¹ using specific families of measure-preserving functions [Lov12, Ch. 7.3], [ES12, Sct. 4.1], [CR20, Sct. 3]. As an example of a non-straight representation, the 2-hypergraphon

$$\mathcal{G}' \stackrel{\text{def}}{=} \{ (x_{\{1\}}, x_{\{2\}}, x_{\{1,2\}}) \mid (x_{\{1\}} + x_{\{2\}} + x_{\{1,2\}}) \bmod 1 \leq p \} \quad (2)$$

¹In the case of a T -on $\mathcal{N}$, we require that this transformation be uniform over all P -ons $\mathcal{N}_P$ forming $\mathcal{N}$.

represents the same limit as the one in (1) but does depend on first-order coordinates.

UCouple $[\ell]$ (Unique ℓ -coupleability) Roughly speaking (the exact definition in the language of open interpretations will be given in Section 2.2), two combinatorial objects ϕ and ψ are uniquely coupleable if any two alignments of these objects on the same ground set (a coupling) give the same object in the combined theory. In that case, this unique coupling can be described by the random alignment, called *independent coupling*, and this allows us to compute the combined object (represented as a flag-algebraic homomorphism) by a very simple formula. For example, quasirandom graphs of density $p \in [0, 1]$ are uniquely coupleable with any 2-coloring of the vertices as well as with the linear ordering. They are *not* uniquely coupleable with themselves, except for the trivial case $p \in \{0, 1\}$. Now, to every combinatorial object ϕ we associate its rank dually to the notion of **Independence**: $\text{rk}(\phi) \leq \ell$ if and only if ϕ has a representation as a T -on $\mathcal{N}$ in which all P -ons $\mathcal{N}_P$ depend *only* on the coordinates x_A with $|A| \leq \ell$. We call an object ϕ *uniquely ℓ -coupleable* if it is uniquely coupleable with all objects ψ such that $\text{rk}(\psi) \leq \ell$.

UInduce $[\ell]$ (Unique ℓ -inducibility) One equivalent way to view the induced subgraph $G|_V$ is this: we first color the vertices into two colors, say, green (corresponding to V) and red. Then instead of removing red vertices, we remove all edges adjacent to at least one red vertex. In this form, it has a perfect generalization in higher dimensions. Namely, we start as in the previous paragraph and consider couplings of a combinatorial object ϕ with an ℓ -hypergraphon ψ (note that $\text{rk}(\psi) \leq \ell$). The unique coupleability requires that for any two such couplings ξ_1 and ξ_2 , we have $\xi_1(M) = \xi_2(M)$ for any model M of the combined theory. Unique inducibility by ψ relaxes this property by requiring that $\xi_1(M) = \xi_2(M)$ holds only for those M that are based on a clique in the hypergraphon ψ . The object ϕ is *uniquely ℓ -inducible* if it is uniquely inducible by any ℓ -hypergraphon ψ .

From the loose formulation of the properties above, one can already see that the first two “naturalness” requirements are satisfied: the formulations are made for arbitrary theories and do not refer to densities of concrete models and their explicit values. As for the third “naturalness” requirement (Theorem 3.3), while the fact that **Independence $[\ell]$** is preserved under open

interpretations follows easily from the general theory, for $\mathbf{UCouple}[\ell]$ and $\mathbf{UInduce}[\ell]$, this will follow from an amalgamation property (Theorem 5.1) that roughly says that couplings can be lifted through open interpretations (Proposition 5.2).

As we mentioned before, the quasirandom k -hypergraph satisfies $\mathbf{Independence}[k-1]$. The situation for asymmetric combinatorial objects is more diverse. For example, the quasirandom tournament satisfies $\mathbf{UCouple}[1]$ but not $\mathbf{Independence}[1]$ and this example can be generalized to higher values of ℓ . One interesting example for unique inducibility is the linear order as it satisfies $\mathbf{UInduce}[\ell]$ for every ℓ without being a trivial object.

All our properties are anti-monotone in ℓ in the sense that for any of the above, we have the implications $P[\ell] \implies P[\ell-1]$ (see Theorem 3.1) and as for relations between the properties (Theorem 3.2), we show that $\mathbf{Independence}[\ell]$ implies $\mathbf{UCouple}[\ell]$ and that $\mathbf{UCouple}[\ell]$ implies² $\mathbf{UInduce}[\ell]$ (see Figure 1).

In terms of separations, we show that no upward implication holds, that is, none of the studied quasirandomness properties with parameter ℓ can imply the same, or for that matter any other, property with parameter $\ell+1$ (Theorem 3.5). As for separations between different families of properties, we show that $\mathbf{UCouple}[\ell]$ does not imply $\mathbf{Independence}[\ell]$ (Theorem 3.6) and $\mathbf{UInduce}[\ell]$ does not imply even $\mathbf{UCouple}[1]$ (Theorem 3.7). We have not been able to extend the latter result to $\mathbf{UCouple}$ vs. $\mathbf{Independence}$, that is to show that $\mathbf{UCouple}[\ell]$ does not imply $\mathbf{Independence}[\ell']$ for a single pair $\ell' < \ell$; in fact these are the only relations involving the three families of properties that we leave open. All these separations are relatively easy when we are working with arbitrary theories, but we show that they still hold even if we restrict ourselves to the theory of k -hypergraphs, for $k \geq \ell+2$ (Theorems 3.8 and 3.9).

Next, we provide the following alternate characterizations (summarized in Theorems 3.10 and 3.11) of these classes.

Weak ℓ -independence Every combinatorial object ϕ can be represented, in a canonical way, by an infinite countable random model $\mathbf{K}$ defined from a collection of independent random variables $(\theta_A)_A$ indexed by finite non-empty subsets of $\mathbb{N}_+$ (see e.g. [CR20, proof of Theorem 3.4]). We say that ϕ is *weakly ℓ -independent* if $\mathbf{K}$ is independent from $(\theta_A \mid$

²This implication is obvious from the definition.

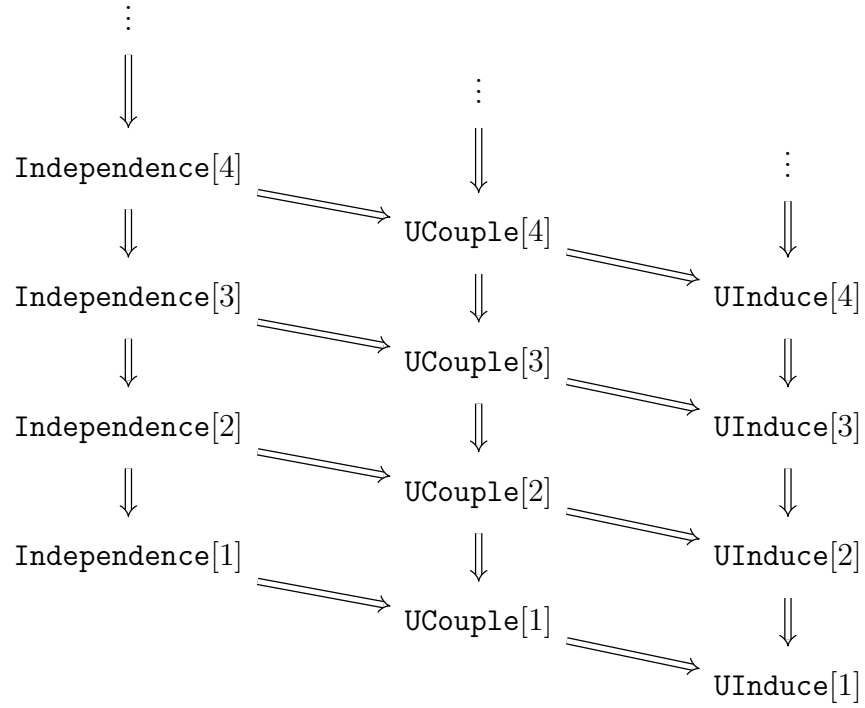


Figure 1: Implications between quasirandomness properties. This is almost a Hasse diagram: only the relations between $\mathbf{UCouple}[\ell]$ and $\mathbf{Independence}[\ell']$ for $\ell' < \ell$ are left open.

$|A| \leq \ell$) as a random variable (full $\text{Independence}[\ell]$ requires this to happen “pointwise”). This weak version of independence turns out to be equivalent to $\text{UCouple}[\ell]$ (Theorem 3.10(iv)).

ℓ -Locality One of the defining properties of the countable random model $\mathbf{K}$ is *locality*: the marginals $(\mathbf{K}|_{V_i} \mid i \in I)$ are mutually independent whenever the collection of finite sets $(V_i)_{i \in I}$ is pairwise disjoint. The notion of *ℓ -locality* strengthens this property to require mutual independence of $(\mathbf{K}|_{V_i} \mid i \in I)$ whenever the collection of finite sets $(V_i)_{i \in I}$ have pairwise intersections of size at most ℓ . It is clear that weak ℓ -independence implies ℓ -locality, but we prove that the converse also holds, hence ℓ -locality is also equivalent to $\text{UCouple}[\ell]$ (Theorem 3.10(vi)).

Symmetric ℓ -locality The notion of *symmetric ℓ -locality* relaxes the notion of ℓ -locality by requiring only mutual independence of the events $(\mathbf{K}|_{V_i} \cong M_i \mid i \in I)$ for all choices of $(V_i)_{i \in I}$ with pairwise intersections of size at most ℓ and all choices of models M_i , i.e., we only care about the submodels $\mathbf{K}|_{V_i}$ up to isomorphism. We show that symmetric ℓ -locality is equivalent to $\text{UInduce}[\ell]$ (Theorem 3.11(iii)).

The right way to view the definitions of unique coupleability and unique inducibility is that each ψ of rank $\leq \ell$ generates a test for the respective property that ϕ has to pass. It is natural to ask for a smaller and more explicit set of universal tests that guarantees each property. We show (Theorem 3.10(ii)) that $\phi \in \text{UCouple}[\ell]$ is equivalent to ϕ being uniquely coupleable with a non-degenerate **quasirandom** ℓ' -hypergraphon $\psi_{\ell', p}$ in every dimension $\ell' \leq \ell$. We further prove (Theorem 3.10(iii)) that it is also equivalent to ϕ being uniquely coupleable with their independent coupling $\psi_{1, p_1} \otimes \dots \otimes \psi_{\ell, p_\ell}$; for the reasons explained right after the statement of the theorem, it does not immediately follow from the previous item (ii). In the particular case $\ell = 1$, this means that the fact that ϕ is uniquely coupleable with a single non-trivial vertex-coloring implies it must also be uniquely coupleable with any rank 1 limit object, such as linear orders, permutations, etc.

Our findings for unique inducibility are by far less conclusive but at least we can show that it is sufficient to consider only hypergraphons ψ with any fixed non-trivial edge density $p \in (0, 1)$ (Theorem 3.11(ii)).

Of all choices of parameters, arguably the most interesting one is when ℓ is exactly one less than the maximum arity k of a predicate of the language. In

the theory of k -hypergraphs the three classes with $\ell = k - 1$ become the same and are satisfied only by the full quasirandom hypergraph, that is, the almost sure limit of the generalization of the Erdős–Rényi model. If we consider general theories of arity at most k , it is not hard to see (Theorem 3.12) that $(k - 1)$ -independent objects are (essentially) quasirandom colored k -hypergraphs. The property **UCouple** $[k - 1]$ in arity at most k corresponds to independent couplings of quasirandom colored k -hypergraphs with generalizations of quasirandom tournaments (Theorem 3.13). The case of unique inducibility is (again) considerably more complicated: **UInduce** $[1]$ in arity 2 corresponds to (essentially) independent couplings of quasirandom colored graphs with an aligned coupling of several biased quasirandom tournaments; this latter aligned coupling is so that all biases are in the same direction. But since this latter proof is very technical and does not seem to easily generalize to arbitrary arities k , we do not include it in the paper.

Finally, let us compare our properties to the known hypergraph quasirandomness properties (Figure 2). In [Tow17], Towsner defined k -hypergraph quasirandomness properties **Disc** $[\mathcal{A}]$ for every antichain $\mathcal{A}$ of non-empty subsets of $[k] \stackrel{\text{def}}{=} \{1, \dots, k\}$ and showed that **Disc** $[\binom{[k]}{\ell}]$ and **Disc** $[\mathcal{A}_\ell]$ are equivalent to **CliqueDisc** $[\ell]$ and **Dev** $[\ell]$ of [LM15b], respectively, where $\mathcal{A}_\ell \stackrel{\text{def}}{=} \{A \in \binom{[k]}{k-\ell} \mid [k - \ell] \subseteq A\}$. It is immediate from definitions that **UInduce** $[\ell]$ implies **CliqueDisc** $[\ell]$ (Theorem 3.14). In terms of separations between our properties and the ones from the literature, we show the strongest separation possible. The strongest **Disc** $[\mathcal{A}]$ property that is not equivalent to full quasirandomness is **Dev** $[k - 1]$ and this does not imply even **UInduce** $[1]$ (Theorem 3.15). In the other direction, the weakest **Disc** $[\mathcal{A}]$ property that is not implied by **CliqueDisc** $[\ell]$ is **Disc** $[\{\ell + 1\}]$ and this is not implied by **Independence** $[\ell]$ (Theorem 3.16).

The paper is organized as follows. In Section 2 we give necessary preliminaries. In Section 3 we formally state our main results. In Section 4, we prove some basic facts that will be used throughout the text. In Section 5, we show that our properties are natural, that is, that they are preserved under open interpretations. In Section 6 we prove the alternative formulations of **UInduce**, and in Section 7 we prove the alternative formulations of **UCouple**. The proofs are done in this slightly reversed order because they are simpler for the unique inducibility; besides, some auxiliary statements we need for that part are later re-used for the unique coupleability. In Section 8, we

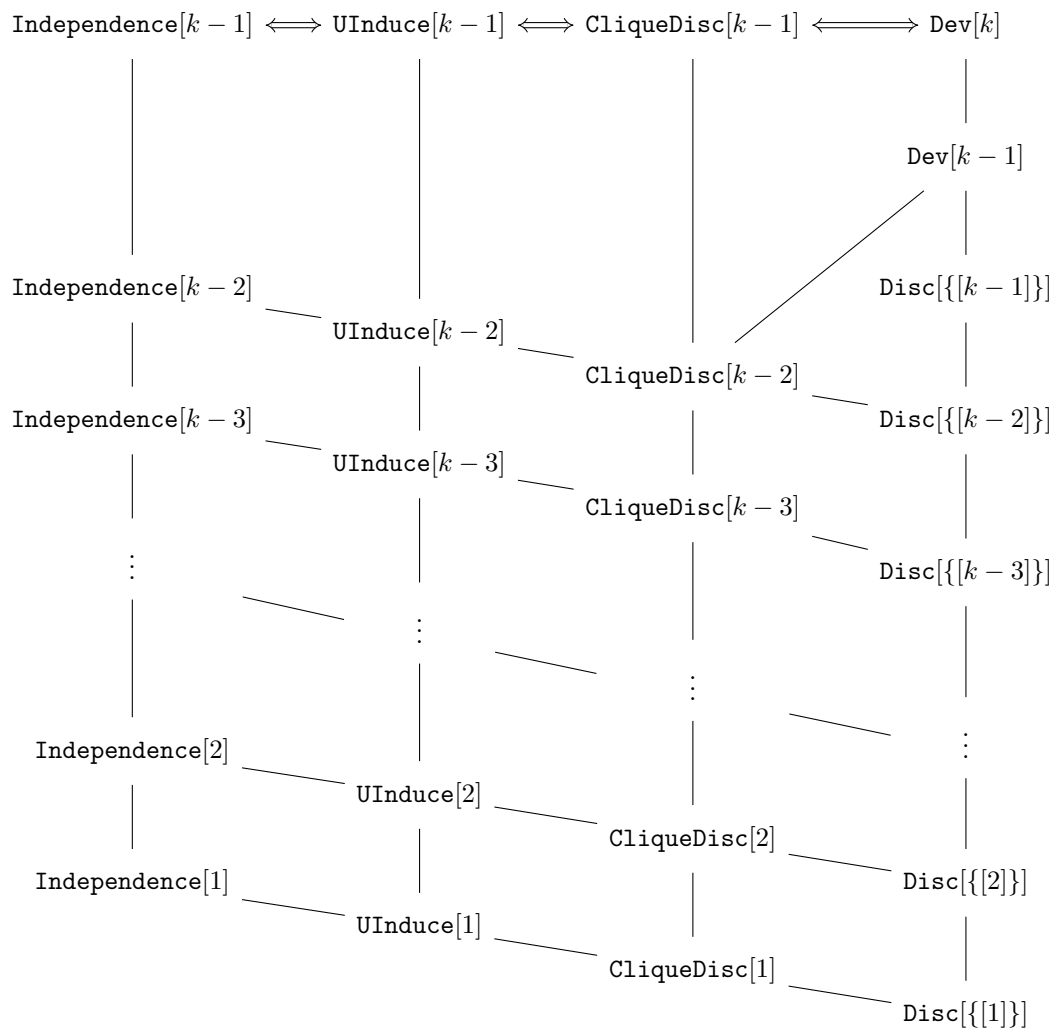


Figure 2: Hasse diagram of quasirandomness hypergraph properties in arity k . The top four equivalent properties represent full quasirandomness.

show separations between different classes of properties. In Section 9, we completely classify the properties **Independence** $[k - 1]$ and **UCouple** $[k - 1]$ when all arities are at most k . The paper is concluded with a few remarks and open problems in Section 10.

2 Preliminaries and notation

Throughout the text, we will use the notation $\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, \dots\}$ for the non-negative integers and $\mathbb{N}_+ \stackrel{\text{def}}{=} \mathbb{N} \setminus \{0\}$ for the positive integers. We also let $[n] \stackrel{\text{def}}{=} \{1, \dots, n\}$ and $(n)_m \stackrel{\text{def}}{=} n(n-1) \cdots (n-m+1)$. The usage of the arrow $\mapsto$ for a function will always presume the function to be injective. For a set V , we let $(V)_\ell$ be the set of all injective functions $\alpha: [\ell] \mapsto V$ and for such an α , we may use the notation α_i for $\alpha(i)$ when convenient. We let $2^V \stackrel{\text{def}}{=} \{A \subseteq V\}$ be the set of all the subsets of V , let $\binom{V}{\ell} \stackrel{\text{def}}{=} \{A \subseteq V \mid |A| = \ell\}$ and let $\binom{V}{>\ell} \stackrel{\text{def}}{=} \{A \subseteq V \mid |A| > \ell\}$. For $V \subseteq \mathbb{N}_+$ and $A \in \binom{V}{\ell}$, we let $\iota_A: [\ell] \mapsto V$ be the function enumerating the set A in the increasing order (so $\text{im}(\iota_A) = A$). We let $r(V)$ be the set of all *finite non-empty* subsets of V and $r(V, \ell) \stackrel{\text{def}}{=} \{A \in r(V) \mid |A| \leq \ell\}$ be the set of all non-empty subsets of V of size at most ℓ . We will be frequently abusing notation by identifying $[n]$ with n , e.g., we will use $r(n, \ell)$ as a shorthand for $r([n], \ell)$. Random variables will always be typed in ***math bold face***. We denote by S_V the group of bijections $V \mapsto V$ so that S_n is the group of permutations on n elements.

2.1 Model theory and limit theory

We will be working in the framework of [CR20], in which combinatorial objects are encoded as models of a canonical theory. We will also be using the same notation as in [CR20] with some small additions.

For a finite relational language $\mathcal{L}$, we let $T_{\mathcal{L}}$ be the pure canonical theory on $\mathcal{L}$, that is, the theory whose axioms are

$$\forall \vec{x}, \left(\bigvee_{1 \leq i < j \leq k(P)} x_i = x_j \right) \rightarrow \neg P(x_1, \dots, x_{k(P)}). \quad (3)$$

for every $P \in \mathcal{L}$. For a canonical theory T and a set V , let $\mathcal{K}_V[T]$ be the set of (labeled) models of T with vertex set V . We use the symbol $\cong$ to indicate

that two models are isomorphic. For $n \in \mathbb{N}_+$, we let $\mathcal{M}_n[T] \stackrel{\text{def}}{=} \mathcal{K}_n[T] / \cong$ be the set of n -element (unlabeled) models up to isomorphism; we also let $\mathcal{M}[T] \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}_+} \mathcal{M}_n[T]$. For $n = |V|$ and $K \in \mathcal{K}_V[T]$, we denote by $[K] \in \mathcal{M}_n[T]$ the isomorphism type of K .

Other important examples of canonical theories include the theory of k -hypergraphs $T_{k\text{-Hypergraph}}$, whose language contains a single predicate E of arity $k(E) \stackrel{\text{def}}{=} k$ and whose axioms are (3) for $P = E$ and

$$\forall \vec{x}, (E(x_1, \dots, x_k) \rightarrow E(x_{\sigma(1)}, \dots, x_{\sigma(k)})) \quad (\sigma \in S_k); \quad (4)$$

the theory of (simple) graphs $T_{\text{Graph}} \stackrel{\text{def}}{=} T_{2\text{-Hypergraph}}$; the theory of (strict) linear orders T_{LinOrder} , whose language contains a single binary predicate $\prec$ with the axioms

$$\begin{aligned} & \forall x, \neg(x \prec x); \\ & \forall \vec{x}, (x_1 \neq x_2 \rightarrow (x_1 \prec x_2 \vee x_2 \prec x_1)); \\ & \forall \vec{x}, (x_1 \prec x_2 \wedge x_2 \prec x_3 \rightarrow x_1 \prec x_3); \end{aligned}$$

and the theory of c -colorings $T_{c\text{-Coloring}}$, whose language contains c unary predicates $\chi_1, \dots, \chi_c$ and that has axioms

$$\begin{aligned} & \forall x, \neg\chi_i(x) \vee \neg\chi_j(x) \quad (1 \leq i < j \leq c); \\ & \forall x, \bigvee_{i \in [c]} \chi_i(x). \end{aligned}$$

Note that $T_{2\text{-Coloring}}$ and $T_{1\text{-Hypergraph}}$ are isomorphic in the category INT (see [CR20, Sct. 2.2]).

Given an atomless complete³ probability space $\Omega = (X, \mathcal{A}, \mu)$, a set V and $\ell \in \mathbb{N}$, we let $\mathcal{E}_{V,\ell}(\Omega) \stackrel{\text{def}}{=} X^{r(V,\ell)}$, equipping it with the completion of the product measure of $|r(V,\ell)|$ copies of μ , which by abuse of notation we also denote by μ (cf. [CR20, Definition 7.3]). Likewise, $\mathcal{E}_V(\Omega) \stackrel{\text{def}}{=} X^{r(V)}$. Given an injective function $\alpha: V_1 \rightarrow V_2$, we define the projection $\alpha^*: \mathcal{E}_{V_2,\ell}(\Omega) \rightarrow \mathcal{E}_{V_1,\ell}(\Omega)$

³ In [CR20, Sct. 7] we carefully considered incomplete spaces as well and drew finer distinctions between various assumptions on them, cf. the discussion in [Lov12, page 218]. It was needed to differentiate between weak theons (satisfying the axioms a.e.) and strong ones (satisfying them everywhere off-diagonal), as well as for removal lemmas. As we prefer to avoid dwelling into these issues in this paper, we make the simplifying assumption of completeness once and for all.

by $\alpha^*(x)_A \stackrel{\text{def}}{=} x_{\alpha(A)}$ (this is consistent with the notation for the projection $\alpha^*: \mathcal{E}_{V_2}(\Omega) \rightarrow \mathcal{E}_{V_1}(\Omega)$ defined analogously in [CR20, Definition 2.19], which we will also use). The spaces used in this paper most often are $([0, 1]^t, \mathcal{L}^t, \lambda^t)$, where $\mathcal{L}^t$ is the σ -algebra of Lebesgue measurable subsets of $[0, 1]^t$ and λ^t is the (t -dimensional) Lebesgue measure; these will be denoted simply by $[0, 1]^t$. When $\Omega = [0, 1]$, we will omit Ω from the notation (e.g., a P -on without reference to any space Ω is assumed to be a measurable subset of $\mathcal{E}_{k(P)} \stackrel{\text{def}}{=} \mathcal{E}_{k(P)}([0, 1])$). For spaces Ω and Ω' , we let $\Omega \times \Omega'$ be the *completion* of the product space. Finally, we will often abuse notation by identifying the spaces $\mathcal{E}_V(\Omega \times \Omega')$ and $\mathcal{E}_V(\Omega) \times \mathcal{E}_V(\Omega')$ via the correspondence $\mathcal{E}_V(\Omega \times \Omega') \ni x \leftrightarrow (y, z) \in \mathcal{E}_V(\Omega) \times \mathcal{E}_V(\Omega')$ given by $y_A \stackrel{\text{def}}{=} (x_A)_1$ and $z_A \stackrel{\text{def}}{=} (x_A)_2$ for every $A \in r(V)$. An analogous identification will be done for products of finitely many spaces.

We also adopt the same conventions as in [CR20]: unless we explicitly say otherwise, all our languages are assumed to be finite first-order relational languages, all our theories are assumed to be canonical (in particular, also universal and we will typically omit universal quantifiers from their axioms) and all our structures are assumed to be canonical (i.e., models of $T_{\mathcal{L}}$, or equivalently, structures K such that $R_P(K) \stackrel{\text{def}}{=} \{\alpha \in V(K)^{k(P)} \mid K \models P(\alpha_1, \dots, \alpha_{k(P)})\}$ is contained in $(V(K))_{k(P)}$ for every $P \in \mathcal{L}$).

Recall that a sequence of finite (unlabeled) models $(N_n)_{n \in \mathbb{N}}$ is *convergent* if $|N_n| < |N_{n+1}|$ and for every fixed finite model M , the limit $\lim_{n \rightarrow \infty} p(M, N_n)$ exists, where $p(M, N)$ denotes the normalized number of unlabeled induced copies of M in N . We will be using three cryptomorphic ways of representing convergent sequences: flag-algebraic homomorphisms [Raz07], theons [CR20, Scts. 3 and 7] and exchangeable arrays [CR20, Definition 5.7]. In this language, a hypergraphon of [ES12] is, up to zero-measure change, a $T_{k\text{-Hypergraph-on}}$ and there is a (not one-to-one) correspondence between graphons W of [LS06] and $T_{\text{Graph-on}}$ s $\mathcal{N}$ that preserves densities given by

$$\begin{aligned} W &\mapsto \{x \in \mathcal{E}_2 \mid x_{\{1,2\}} < W(x_{\{1\}}, x_{\{2\}})\} \\ W_{\mathcal{N}} &\leftarrow \mathcal{N}, \end{aligned}$$

where

$$W_{\mathcal{N}}(x_{\{1\}}, x_{\{2\}}) \stackrel{\text{def}}{=} \lambda(\{x_{\{1,2\}} \mid (x_{\{1\}}, x_{\{2\}}, x_{\{1,2\}}) \in \mathcal{N}\}). \quad (5)$$

Furthermore, for $M \in \mathcal{M}[T]$ we let

$$\langle M \rangle \stackrel{\text{def}}{=} \frac{|\text{Aut}(M)|}{|M|!} M \quad (6)$$

denote⁴ the element of the flag algebra $\mathcal{A}[T]$ encoding the *labeled* (induced) density of M .

The main theorem of dense limit theory says that positive homomorphisms, theons and local exchangeable arrays all encode convergent sequences.

Theorem 2.1 ([LS06, Raz07], [CR20, Theorem 6.3] see also [CR20, Sct. 7]). *Fix an atomless complete probability space Ω and consider the following objects for a theory T .*

- i. *A convergent sequence $(N_n)_{n \in \mathbb{N}}$ of models of T .*
- ii. *A positive homomorphism $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$.*
- iii. *A T -on $\mathcal{N}$ over Ω .*
- iv. *A local exchangeable array $\mathbf{K}$ supported on models of T .*

The objects above are cryptomorphic in the sense that given an instance of one of them, one can “explicitly” construct instances of the others that satisfy the following for every $M \in \mathcal{M}[T]$:

$$\lim_{n \rightarrow \infty} p(M, N_n) = \phi(M) = \phi_{\mathcal{N}}(M) = \mathbb{P}[\mathbf{K}|_{[|M|]} \cong M].$$

One of the (easy) directions of the cryptomorphism above will be of particular importance to us, namely, how to construct a local exchangeable array $\mathbf{K}$ from a given T -on $\mathcal{N}$ over $\Omega = (X, \mathcal{A}, \mu)$. Intuitively, the only thing we have to do is to independently sample countably many points from our theon. Formally, let $\boldsymbol{\theta} = (\boldsymbol{\theta}_A)_{A \in r(\mathbb{N}_+)}$ be picked in $\mathcal{E}_{\mathbb{N}_+}(\Omega)$ according to [the product measure] μ , that is, each $\boldsymbol{\theta}_A$ picked in X according to μ independently of all other coordinates. The *exchangeable array $\mathbf{K}$ corresponding to $\mathcal{N}$ with respect to $\boldsymbol{\theta}$* is defined by

$$V(\mathbf{K}) \stackrel{\text{def}}{=} \mathbb{N}_+, \quad R_P(\mathbf{K}) \stackrel{\text{def}}{=} \{\alpha \in (\mathbb{N}_+)_{k(P)} \mid \alpha^*(\boldsymbol{\theta}) \in \mathcal{N}_P\} \quad (7)$$

⁴Note that if we think of M as a flag algebra type, then this notation is compatible with [Raz07, Definition 8]. But in this paper, like in [CR20], we try to avoid flag algebras in non-trivial types.

and we have $\phi_{\mathcal{N}}(M) = \mathbb{P}[\mathbf{K}|_{\llbracket M \rrbracket} \cong M]$ for every $M \in \mathcal{M}[T]$ (see [CR20, proof of Theorem 3.4]).

Once we capture combinatorial objects as models of canonical theories, local combinatorial constructions are then captured by open interpretations (see [CR20, Sect. 2.2]) in the sense that if $I: T_1 \rightsquigarrow T_2$ is an open interpretation and K is a model of T_2 , there is a naturally defined model $I(K)$ of T_1 given by $V(I(K)) \stackrel{\text{def}}{=} V(K)$ and $R_P(I(K)) \stackrel{\text{def}}{=} \{\alpha \in (V(K))_{k(P)} \mid K \models I(P)(\alpha_1, \dots, \alpha_{k(P)})\}$. The simplest but most important type of open interpretations are the *structure-erasing* interpretations, which are open interpretations of the form $I: T_1 \rightsquigarrow T_1 \cup T_2$, where $T_1 \cup T_2$ is the disjoint union of the theories T_1 and T_2 . They act identically on the language of T_1 , and the corresponding combinatorial construction corresponds to erasing all information of T_2 . Convergent sequences behave very well with respect to open interpretations, namely, if $(N_n)_{n \in \mathbb{N}}$ is a convergent sequence of models of T_2 , then $(I(N_n))_{n \in \mathbb{N}}$ is a convergent sequence of models of T_1 . This behavior is translated to operations on the limit objects of Theorem 2.1. Namely, if $\phi \in \text{Hom}^+(\mathcal{A}[T_2], \mathbb{R})$, the T_2 -on $\mathcal{N}$ and the array $\mathbf{K}$ correspond to a convergent sequence $(N_n)_{n \in \mathbb{N}}$ of models of T_2 under Theorem 2.1, then $\phi^I \stackrel{\text{def}}{=} \phi \circ \pi^I$ (where π^I is $\pi^{(U, I)}$ in [Raz07, Definition 4 and Theorem 2.6] when $U(x)$ is $x = x$; see also [CR20, Theorem 2.14]), $I(\mathcal{N})$ given by $I(\mathcal{N})_P \stackrel{\text{def}}{=} T(I(P), \mathcal{N})$ (see [CR20, Definition 3.5]) and $I(\mathbf{K})$ are limit objects corresponding to $(I(N_n))_{n \in \mathbb{N}}$ for Theorem 2.1 (see [CR20, Remark 6]).

Finally, let us denote the identity interpretation of a theory T by $\text{id}_T: T \rightsquigarrow T$ and for interpretations $I: T_1 \rightsquigarrow T_3$ and $J: T_2 \rightsquigarrow T_4$, we denote by $I \cup J: T_1 \cup T_2 \rightsquigarrow T_3 \cup T_4$ the amalgamation interpretation that acts as I on T_1 and acts as J on T_2 .

2.2 Quasirandomness properties

In this section we formalize all notions of quasirandomness presented in the introduction.

Definition 2.2 (rank and independence). The *rank* of a peon $\mathcal{N} \subseteq \mathcal{E}_k(\Omega)$ over $\Omega = (X, \mathcal{A}, \mu)$, denoted $\text{rk}(\mathcal{N})$, is the minimum $r \in \mathbb{N}$ such that $\mathcal{N}$ can be written as $\mathcal{N} = \mathcal{H} \times X^{(\begin{smallmatrix} [k] \\ >r \end{smallmatrix})}$ for some $\mathcal{H} \subseteq \mathcal{E}_{k,r}(\Omega)$. The *rank* of an Euclidean structure $\mathcal{N}$ is the maximum rank $\text{rk}(\mathcal{N})$ of its peons.

Dually, for $\ell \in \mathbb{N}$, a peon $\mathcal{N} \subseteq \mathcal{E}_k(\Omega)$ is called *ℓ -independent* if it can be

written as $\mathcal{N} = \mathcal{E}_{k,\ell}(\Omega) \times \mathcal{H}$ for some $\mathcal{H} \subseteq X^{([k]_{>\ell})}$ and an Euclidean structure is called ℓ -independent if all of its peons are ℓ -independent.

For $\ell \in \mathbb{N}$, an Euclidean structure $\mathcal{N}$ on $\mathcal{L}$ over Ω is *weakly ℓ -independent* if the exchangeable array $\mathbf{K}$ corresponding to $\mathcal{N}$ with respect to $\boldsymbol{\theta}$ picked in $\mathcal{E}_{\mathbb{N}_+}(\Omega)$ according to μ (see (7)) is independent from $(\boldsymbol{\theta}_A \mid A \in r(\mathbb{N}_+, \ell))$ as a random variable.

Given $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$, the *rank* of ϕ , denoted $\text{rk}(\phi)$, is the minimum rank of a T -on $\mathcal{N}$ such that $\phi_{\mathcal{N}} = \phi$. Dually, we say $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ is ℓ -independent (resp., *weakly ℓ -independent*) if there exists an ℓ -independent (resp., *weakly ℓ -independent*) T -on $\mathcal{N}$ such that $\phi_{\mathcal{N}} = \phi$. We will refer to the former property as **Independence** $[\ell]$ but we do not introduce any special notation for weak independence as it will be shown to be equivalent to another property below.

Definition 2.3 (couplings). Given canonical theories $T_1, \dots, T_t$ and $\phi_i \in \text{Hom}^+(\mathcal{A}[T_i], \mathbb{R})$ ($i \in [t]$), a *coupling* of $\phi_1, \dots, \phi_t$ is a positive homomorphism $\xi \in \text{Hom}^+(\mathcal{A}[\bigcup_{i \in [t]} T_i], \mathbb{R})$ such that $\xi^{I_i} = \phi_i$ for every $i \in [t]$, where $I_i: T_i \rightsquigarrow \bigcup_{j \in [t]} T_j$ is the structure-erasing interpretation.

The most important coupling is the independent coupling defined below. In the finite world, the independent coupling of limits of sequences $(N_n^i)_{n \in \mathbb{N}}$ with $V(N_n^i) = V(N_n^j)$ corresponds to the almost sure limit of the random sequence $(\mathbf{N}_n)_{n \in \mathbb{N}}$ where $\mathbf{N}_n$ is obtained by first randomly permuting the vertices of each N_n^i uniformly and independently and coupling the result.

Definition 2.4 (independent coupling, semantic version). For $i \in [t]$, let $\mathcal{N}^i$ be a T_i -on over Ω_i . The *independent coupling* of $\mathcal{N}^1, \dots, \mathcal{N}^t$ is the $(\bigcup_{i \in [t]} T_i)$ -on $\mathcal{N}^1 \otimes \dots \otimes \mathcal{N}^t$ over $\prod_{i \in [t]} \Omega_i$ defined by

$$(\mathcal{N}^1 \otimes \dots \otimes \mathcal{N}^t)_P \stackrel{\text{def}}{=} \left\{ x \in \prod_{j \in [t]} \mathcal{E}_{k(P)}(\Omega_j) \mid \pi_i(x) \in \mathcal{N}_P^i \right\},$$

whenever P is in the language of T_i and where π_i denotes the natural projection on the i -th coordinate.

Definition 2.5 (independent coupling, syntactic version). For $i \in [t]$, let $\phi_i \in \text{Hom}^+(\mathcal{A}[T_i], \mathbb{R})$. The *independent coupling* $\phi_1 \otimes \dots \otimes \phi_t \in \text{Hom}^+(\mathcal{A}[\bigcup_{i \in [t]} T_i], \mathbb{R})$ of $\phi_1, \dots, \phi_t$ is defined by

$$(\phi_1 \otimes \dots \otimes \phi_t)(\langle M \rangle) \stackrel{\text{def}}{=} \prod_{i \in [t]} \phi_i(\langle I_i(M) \rangle),$$

for every $M \in \mathcal{M}[\bigcup_{i \in [t]} T_i]$, where $I_i: T_i \rightsquigarrow \bigcup_{j \in [t]} T_j$ is the structure-erasing interpretation.

These two definitions are obviously consistent: if $\mathcal{N}^i$ is a T_i -on over Ω_i such that $\phi_{\mathcal{N}^i} = \phi_i$ ($i \in [t]$), then $(\phi_1 \otimes \cdots \otimes \phi_t) = \phi_{\mathcal{N}^1 \otimes \cdots \otimes \mathcal{N}^t}$. In particular, this implies that $\phi_1 \otimes \cdots \otimes \phi_t \in \text{Hom}^+(\mathcal{A}[\bigcup_{i \in [t]} T_i], \mathbb{R})$ (which can be also verified by a direct computation).

Definition 2.6 (unique coupleability and inducibility). We say that $\phi_1, \dots, \phi_t$ are *uniquely coupleable* if the independent coupling is their only coupling. For $\ell \in \mathbb{N}$, we say that $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ is *uniquely ℓ -coupleable* if for every theory T' and every $\psi \in \text{Hom}^+(\mathcal{A}[T'], \mathbb{R})$ with $\text{rk}(\psi) \leq \ell$, ϕ and ψ are uniquely coupleable. We will be using the abbreviation $\text{UCouple}[\ell]$ for this property.

Given $\ell \in \mathbb{N}_+$, $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ and $\psi \in \text{Hom}^+(\mathcal{A}[T_{\ell\text{-Hypergraph}}], \mathbb{R})$, we say that ϕ is *uniquely inducible* by ψ if for any coupling ξ of ϕ and ψ and for every $M \in \mathcal{M}[T \cup T_{\ell\text{-Hypergraph}}]$ such that $I(M)$ is a complete ℓ -hypergraph, we have $\xi(M) = (\phi \otimes \psi)(M)$, where $I: T_{\ell\text{-Hypergraph}} \rightsquigarrow T \cup T_{\ell\text{-Hypergraph}}$ is the structure-erasing interpretation. We say that ϕ is *uniquely ℓ -inducible* if it is uniquely inducible by every $\psi \in \text{Hom}^+(\mathcal{A}[T_{\ell\text{-Hypergraph}}], \mathbb{R})$, and we will be using the abbreviation $\text{UInduce}[\ell]$. For completeness, we declare every ϕ to satisfy $\text{UInduce}[0]$.

Remark 1. Since $T_{1\text{-Hypergraph}} \cong T_{2\text{-Coloring}}$, for $\ell = 1$ we prefer to work with the following equivalent formulation of $\text{UInduce}[1]$ that can be deduced from this isomorphism. $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ is uniquely inducible by $\psi \in \text{Hom}^+(\mathcal{A}[T_{2\text{-Coloring}}], \mathbb{R})$ if for any coupling ξ of ϕ and ψ and for every $M \in \mathcal{M}[T \cup T_{2\text{-Coloring}}]$ such that $R_{\chi_1}(M) = V(M)$, we have $\xi(M) = (\phi \otimes \psi)(M)$. Then ϕ is uniquely 1-inducible if it is uniquely inducible by every $\psi \in \text{Hom}^+(\mathcal{A}[T_{2\text{-Coloring}}], \mathbb{R})$.

Also, as we will see below (Theorem 3.1), $\text{UInduce}[\ell]$ implies $\text{UInduce}[\ell']$ for any $\ell' \leq \ell$. Hence, we could have equivalently required in this definition unique inducibility by every $\psi \in \text{Hom}^+(\mathcal{A}[T_{\ell'\text{-Hypergraph}}], \mathbb{R})$ with $\ell' \leq \ell$.

These three properties are central to our paper. If P is any of them, we will say interchangeably that ϕ satisfies $P[\ell]$ or that $\phi \in P[\ell]$.

Definition 2.7 (locality). Let $\mathcal{N}$ be a T -on over $\Omega = (X, \mathcal{A}, \mu)$ and let $\mathbf{K}$ be the exchangeable array corresponding to $\mathcal{N}$ with respect to θ picked in $\mathcal{E}_{\mathbb{N}_+}(\Omega)$ according to μ (see (7)).

We say that $\mathcal{N}$ is *ℓ -local* if for every collection $(V_i)_{i \in I}$ of finite subsets of

$\mathbb{N}_+$ with pairwise intersections of size at most ℓ , the marginals $(\mathbf{K}|_{V_i} \mid i \in I)$ are mutually independent.

We say that $\mathcal{N}$ is *symmetrically ℓ -local* if for every collection $(V_i)_{i \in I}$ of finite subsets of $\mathbb{N}_+$ with pairwise intersections of size at most ℓ , the random variables $([\mathbf{K}|_{V_i}] \mid i \in I)$ (recall that $[K]$ is the isomorphism type of K) are mutually independent.

We say that $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ is *ℓ -local* (resp., *symmetrically ℓ -local*) if there exists an ℓ -local (resp., symmetrically ℓ -local) T -on $\mathcal{N}$ such that $\phi = \phi_{\mathcal{N}}$.

Note that both the notions of 0-locality and symmetric 0-locality coincide with the notion of locality for $\mathbf{K}$ (see [CR20, Definition 5.12]). Besides, it is very easy to give an explicit purely syntactic description of both locality and symmetric locality in the style of Definition 2.5; this in particular implies that for an ℓ -local (resp., symmetrically ℓ -local) $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$, every T -on $\mathcal{N}$ with $\phi = \phi_{\mathcal{N}}$ must necessarily be ℓ -local (resp., symmetrically ℓ -local).

Finally, let us state the properties $\text{CliqueDisc}[\ell]$ and $\text{Disc}[\mathcal{A}]$ in the limit language.

Definition 2.8. Let $K_n^{(t)} \in \mathcal{M}_n[T_{t\text{-Hypergraph}}]$ be the complete t -uniform hypergraph on n vertices and let $\rho_t \stackrel{\text{def}}{=} K_t^{(t)}$. Let $\phi \in \text{Hom}^+(\mathcal{A}[T_{k\text{-Hypergraph}}], \mathbb{R})$ and $\ell \in [k]$.

We say that ϕ satisfies $\text{CliqueDisc}[\ell]$ ([LM15b]) if for every $\psi \in \text{Hom}^+(\mathcal{A}[T_{\ell\text{-Hypergraph}}], \mathbb{R})$ and every coupling ξ of ϕ and ψ , we have

$$\xi(K_k^{(k,\ell)}) = \phi(\rho_k)\psi(K_k^{(\ell)}),$$

where $K_k^{(k,\ell)} \in \mathcal{M}_k[T_{k\text{-Hypergraph}} \cup T_{\ell\text{-Hypergraph}}]$ is the model obtained by aligning ρ_k and $K_k^{(\ell)}$ (i.e., the model of size k that is a complete hypergraph in both theories).

Given an antichain $\mathcal{A} \subseteq r(k)$, let $\mathcal{L}_{\mathcal{A}}$ be the language containing one predicate symbol P_A of arity $k(P_A) \stackrel{\text{def}}{=} |A|$ for every $A \in \mathcal{A}$. We say that ϕ satisfies $\text{Disc}[\mathcal{A}]$ ([Tow17, AHCH⁺18]) if for every $\psi \in \text{Hom}^+(\mathcal{A}[T_{\mathcal{L}_{\mathcal{A}}}], \mathbb{R})$ and every coupling ξ of ϕ and ψ , if $\mathbf{K}$ is the exchangeable array in $\mathcal{K}_{\mathbb{N}_+}[T_{k\text{-Hypergraph}} \cup T_{\mathcal{L}_{\mathcal{A}}}]$ associated with ξ , then we have

$$\begin{aligned} \mathbb{P}[(1, \dots, k) \in R_E(\mathbf{K}) \wedge \forall A \in \mathcal{A}, \iota_A \in R_{P_A}(\mathbf{K})] \\ = \phi(\rho_k) \cdot \mathbb{P}[\forall A \in \mathcal{A}, \iota_A \in R_{P_A}(\mathbf{K})], \end{aligned}$$

that is, the events $(1, 2, \dots, k) \in R_E(\mathbf{K})$ and $\forall A \in \mathcal{A}, \iota_A \in R_{P_A}(\mathbf{K})$ are independent.

In [Tow17], the definition of $\text{Disc}[\mathcal{A}]$ further requires symmetry of the predicate symbols P_A , but it was shown in [AHCH⁺18] that this condition can be dropped.

2.3 Useful theories and objects

In this final preliminary subsection, we define some theories and limit objects that are necessary to formally state some of our main results.

We will denote by ψ_{lin} the (unique) element of $\text{Hom}^+(\mathcal{A}[T_{\text{LinOrder}}], \mathbb{R})$. As for the rest, we start with a very general definition (that nonetheless will be used in full generality in Theorem 3.13) and then derive all others as special cases.

For $c \geq 2$, let $\Pi_c \stackrel{\text{def}}{=} \{p = (p_i)_{i=1}^c \in (0, 1)^c \mid \sum_{i=1}^c p_i = 1\}$ be the interior of the standard $(c - 1)$ -dimensional simplex. Also, given $x \in \mathcal{E}_n$, let $\sigma_x \in S_n$ be the unique permutation such that $x_{\{\sigma_x^{-1}(1)\}} < \dots < x_{\{\sigma_x^{-1}(n)\}}$ when the coordinates $(x_{\{i\}} \mid i \in [n])$ are distinct, and define it arbitrarily otherwise.

Definition 2.9 (S_k -action theories). Let $k \in \mathbb{N}_+$, let $\mathcal{L}$ be a language containing only predicate symbols of arity exactly k , let $\Theta: S_k \times \mathcal{L} \rightarrow \mathcal{L}$ be a (left) action of S_k on $\mathcal{L}$ and write $\sigma \cdot P \stackrel{\text{def}}{=} \Theta(\sigma, P)$. The canonical theory T_Θ is defined as the theory over $\mathcal{L}$ with axioms

$$\left(\bigwedge_{1 \leq i < j \leq k} x_i \neq x_j \right) \equiv \left(\bigvee_{P \in \mathcal{L}} P(x_1, \dots, x_k) \right); \quad (8)$$

$$P(x_{\sigma(1)}, \dots, x_{\sigma(k)}) \equiv (\sigma \cdot P)(x_1, \dots, x_k) \quad (P \in \mathcal{L}, \sigma \in S_k); \quad (9)$$

$$\neg P(x_1, \dots, x_k) \vee \neg P'(x_1, \dots, x_k) \quad (P, P' \in \mathcal{L}, P \neq P'). \quad (10)$$

Given a Θ -invariant $p = (p_P)_{P \in \mathcal{L}} \in [0, 1]^\mathcal{L}$ with $\sum_{P \in \mathcal{L}} p_P = 1$, the (Θ, p) -*quasirandom homomorphism* is the homomorphism $\psi_{\Theta, p} \in \text{Hom}^+(\mathcal{A}[T_\Theta], \mathbb{R})$ corresponding to picking at random for each k -set A , independently of other k -sets, an orbit $O \subseteq \mathcal{L}$ of the action Θ with probability $\sum_{P \in O} p_P$ then uniformly at random choosing an S_k -equivariant assignment of the k -tuples with image A to the elements of O . A T_Θ -on $\mathcal{N}^Z$ representing $\psi_{\Theta, p}$ is given by⁵

$$\mathcal{N}_P^Z \stackrel{\text{def}}{=} \{x \in \mathcal{E}_k \mid x_{[k]} \in Z_{\sigma_x \cdot P}\} \quad (P \in \mathcal{L}), \quad (11)$$

⁵We will check that all axioms of T_Θ are satisfied and provide an alternate syntactic description as part of Proposition 9.1.

where $Z = (Z_P)_{P \in \mathcal{L}}$ is a measurable partition of $[0, 1]$ with $\lambda(Z_P) = p_P$ ($P \in \mathcal{L}$).

Let us now note a few special cases that will play an active role in our paper.

Definition 2.10 (*c-colored k-hypergraphs*). Let $\mathcal{L} = \{E_1, \dots, E_c\}$ and assume that the action Θ is trivial. In that case we will denote the theory T_Θ by $T_{c,k}$ and call it the *theory of c-colored k-hypergraphs*. The (Θ, p) -quasirandom homomorphism will be called *quasirandom c-colored k-hypergraphon with densities p* and denoted by $\psi_{k,p}$.

Definition 2.11 (*quasirandom k-hypergraphons*). Let us further specify $c = 2$ in the previous definition. Since E_2 is the negation of E_1 and hence can be safely removed, the theory T_Θ is isomorphic to $T_{k\text{-Hypergraph}}$. For $p \in (0, 1)$, the $(\Theta, (p, 1 - p))$ -quasirandom homomorphism is called the *quasirandom k-hypergraphon of density p*; it will also be denoted by $\psi_{k,p}$.

Definition 2.12 (*Colorings*). Letting instead $k = 1$ in Definition 2.9, and keeping the action Θ trivial, we see that T_Θ is naturally isomorphic to the theory $T_{c\text{-Coloring}}$. The quasirandom object will be called *c-coloring with densities p*, $p \in \Pi_c$, and denoted by $\psi_p \in \text{Hom}^+(\mathcal{A}[T_{c\text{-Coloring}}], \mathbb{R})$. For $c = 2$ and $p \in (0, 1)$, $\psi_{(p, 1-p)}$ will be often abbreviated to ψ_p (which, in view of Remark 1, is also the same as $\psi_{1,p} \in \text{Hom}^+(\mathcal{A}[T_{1\text{-Hypergraph}}], \mathbb{R})$).

Definition 2.13 (*k-tournaments*). Let now $\mathcal{L} = \{E_1, E_2\}$ and $k \geq 2$, but this time the action Θ is not trivial but instead given by the sign homomorphism $\text{sgn}: S_k \rightarrow S_2$. Then the only Θ -invariant p is $p_1 = p_2 = 1/2$ and, as in the case of hypergraphons, we can exclude E_2 from the theory. We call it the theory of *k-tournaments* and denote by $T_{k\text{-Tournament}}$; intuitively, this theory corresponds to choosing one of the two possible orientations for every k -set. The quasirandom object $\psi_{\Theta, (1/2, 1/2)}$ will then be called the *quasirandom k-tournamon* and denoted by ψ_k ; thus, $\psi_k \in \text{Hom}^+(\mathcal{A}[T_{k\text{-Tournament}}], \mathbb{R})$, and ψ_2 is the ordinary quasirandom tournamon.

3 Main results

In this section we present the main results. We remark that some of these results follow trivially from definitions and we will point these out as we go along.

Theorem 3.1. *The properties **Independence**, **UCouple** and **UInduce** are anti-monotone in the sense that $P[\ell] \implies P[\ell - 1]$.*

For **Independence** and **UCouple**, this theorem trivially follows from definitions. Even though it is possible to give an ad hoc proof that **UInduce** is also anti-monotone, this follows trivially from its equivalence with symmetric locality (Theorem 3.11 below) and the fact that symmetric locality is trivially anti-monotone.

Theorem 3.2. *For any $\ell \in \mathbb{N}$, $\text{Independence}[\ell] \implies \text{UCouple}[\ell] \implies \text{UInduce}[\ell]$.*

The second implication follows trivially from the definitions.

The next theorem concerns preservation of properties under open interpretations.

Theorem 3.3 (Naturality). *Let $I: T_1 \rightsquigarrow T_2$ be an open interpretation and let $\ell \in \mathbb{N}$. The following hold for any $\phi \in \text{Hom}^+(\mathcal{A}[T_2], \mathbb{R})$.*

- i. *If ϕ is uniquely coupleable with some $\psi \in \text{Hom}^+(\mathcal{A}[T_1], \mathbb{R})$, then ϕ^I is uniquely coupleable with ψ .*
- ii. *If $\phi \in \text{Independence}[\ell]$, then $\phi^I \in \text{Independence}[\ell]$.*
- iii. *If $\phi \in \text{UCouple}[\ell]$, then $\phi^I \in \text{UCouple}[\ell]$.*
- iv. *If $\phi \in \text{UInduce}[\ell]$, then $\phi^I \in \text{UInduce}[\ell]$.*

Item (ii) follows trivially from the definition of $I(\mathcal{N})$ applied to an ℓ -independent T_2 -on $\mathcal{N}$ such that $\phi = \phi_{\mathcal{N}}$. Note also that item (iii) follows trivially from item (i). Furthermore, applying this theorem to the axiom-adding interpretation $I: T_{\mathcal{L}} \rightsquigarrow T$, where $\mathcal{L}$ is the language of T , we see that all our main notions do not depend on non-logical axioms. Nonetheless, using theories and theons (as opposed to arbitrary Euclidean structures) helps to better orient ourselves and put many of the results in the “right” focus.

The next theorem says that both **Independence** and **UCouple** are preserved under independent couplings.

Theorem 3.4. *Let $\phi_1 \in \text{Hom}^+(\mathcal{A}[T_1], \mathbb{R})$ and $\phi_2 \in \text{Hom}^+(\mathcal{A}[T_2], \mathbb{R})$. The following hold for $\ell \in \mathbb{N}$.*

- i. *If $\phi_1, \phi_2 \in \text{Independence}[\ell]$, then $\phi_1 \otimes \phi_2 \in \text{Independence}[\ell]$.*

ii. If $\phi_1, \phi_2 \in \mathbf{UCouple}[\ell]$, then $\phi_1 \otimes \phi_2 \in \mathbf{UCouple}[\ell]$.

Remarkably, this is not true for $\mathbf{UInduce}$, and a good example is provided by the quasirandom permuton (see the end of this section).

The next five theorems concern separations between properties, either allowing general theories or restricted to the theory of hypergraphs.

Theorem 3.5. *$\mathbf{Independence}[\ell]$ does not imply $\mathbf{UInduce}[\ell + 1]$, not even when restricted to the theory of k -hypergraphs as long as $k > \ell$.*

In fact, this theorem is a consequence of Theorems 3.14 and 3.16 below.

The following two theorems are included since the separating objects are quite natural and explicit and the proofs are simpler. But in a sense they will be superseded by Theorems 3.8 and 3.9.

Theorem 3.6. *For every $\ell \in \mathbb{N}_+$, the quasirandom $(\ell + 1)$ -tournamon $\psi_{\ell+1}$ satisfies $\mathbf{UCouple}[\ell]$ but does not satisfy $\mathbf{Independence}[\ell]$.*

Theorem 3.7. *The linear order $\psi_{\text{lin}} \in \text{Hom}^+(\mathcal{A}[T_{\text{LinOrder}}], \mathbb{R})$ satisfies $\mathbf{UInduce}[\ell]$ for every $\ell \in \mathbb{N}$ but does not satisfy $\mathbf{UCouple}[1]$.*

Theorem 3.8. *For $\ell \geq 1$, there exists $\phi \in \text{Hom}^+(\mathcal{A}[T_{(\ell+2)\text{-Hypergraph}}], \mathbb{R})$ satisfying $\mathbf{UCouple}[\ell]$ but not satisfying $\mathbf{Independence}[\ell]$.*

Theorem 3.9. *For $\ell \geq 1$ odd, there exists $\phi \in \text{Hom}^+(\mathcal{A}[T_{(\ell+2)\text{-Hypergraph}}], \mathbb{R})$ satisfying $\mathbf{UInduce}[\ell]$ but not satisfying $\mathbf{UCouple}[1]$.*

The next theorem lists several properties that are equivalent to $\mathbf{UCouple}[\ell]$. These include both alternative formulations and complete sets of tests for unique coupleability.

Theorem 3.10 (Characterization of $\mathbf{UCouple}$). *Let $\ell \in \mathbb{N}_+$. The following are equivalent for $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$.*

- i. $\phi \in \mathbf{UCouple}[\ell]$.
- ii. For every $\ell' \in [\ell]$, there exists $p \in (0, 1)$ such that ϕ is uniquely coupleable with the quasirandom ℓ' -hypergraphon $\psi_{\ell', p}$.
- iii. There exist $p_1, \dots, p_\ell \in (0, 1)$ such that ϕ is uniquely coupleable with the independent coupling $\psi_{1, p_1} \otimes \dots \otimes \psi_{\ell, p_\ell}$ of the quasirandom ℓ' -hypergraphons $\psi_{\ell', p_{\ell'}}$ for $\ell' \in [\ell]$.
- iv. ϕ is weakly ℓ -independent.

- v. Every T -on $\mathcal{N}$ with $\phi_{\mathcal{N}} = \phi$ is weakly ℓ -independent.
- vi. ϕ is ℓ -local.
- vii. $\phi \otimes \psi_{\text{lin}}$ satisfies $\mathbf{UInduce}[\ell]$.

Note that since ℓ' -hypergraphons have rank at most ℓ' , a posteriori, we can also strengthen items (ii) and (iii) by replacing existential quantifiers on $p, p_1, \dots, p_\ell$ with universal ones. Also, since the linear order has rank 1, a posteriori, we can strengthen item (vii) to say that *every* coupling of ϕ with the linear order satisfies $\mathbf{UInduce}[\ell]$. In the actual proof of the implication (ii) $\implies$ (i) (that, arguably, is our technically most difficult result), we go in the opposite direction and painstakingly “bootstrap” the premise in (ii) to the unique coupleability with increasingly larger families of objects.

Let us also point out that, given Theorem 3.4(ii), one might expect that, in general, if each one of $\psi_1, \dots, \psi_t$ is uniquely coupleable with a given ϕ , then the same should hold for their independent coupling $\psi_1 \otimes \dots \otimes \psi_t$; this would immediately give (ii) $\implies$ (iii) in Theorem 3.10. However, this question has turned out surprisingly difficult in full generality (see Section 10 for a discussion).

The next, more modest, theorem provides properties equivalent to $\mathbf{UInduce}[\ell]$.

Theorem 3.11 (Characterization of $\mathbf{UInduce}$). *The following are equivalent for $\ell \in \mathbb{N}_+$ and $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$.*

- i. $\phi \in \mathbf{UInduce}[\ell]$.
- ii. There exists $p \in (0, 1)$ such that ϕ is uniquely inducible by every $\psi \in \text{Hom}^+(\mathcal{A}[T_{\ell\text{-Hypergraph}}], \mathbb{R})$ with $\psi(\rho_\ell) = p$.
- iii. ϕ is symmetrically ℓ -local.

The next two theorems completely classify $\mathbf{Independence}[k-1]$ and $\mathbf{UCouple}[k-1]$ when all arities are at most k . These can be thought of as analogues of full quasirandomness for these families of properties.

Theorem 3.12. *Let $k \in \mathbb{N}_+$ and suppose that $k(P) \leq k$ for all $P \in \mathcal{L}$. Let T be a theory over $\mathcal{L}$ and $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$. Then $\phi \in \mathbf{Independence}[k-1]$ if and only if there exist $c \in \mathbb{N}_+$, $p \in \Pi_c$ and an open interpretation $I: T \rightsquigarrow T_{c,k}$ such that $\phi = \psi_{k,p}^I$.*

Theorem 3.13. *Let $k \in \mathbb{N}_+$ and suppose that $k(P) \leq k$ for all $P \in \mathcal{L}$. Let T be a theory over $\mathcal{L}$ and $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$. Then $\phi \in \text{UCouple}[k-1]$ if and only if there exists a language $\mathcal{L}'$ whose predicate symbols have arity exactly k , an action $\Theta: S_k \times \mathcal{L}' \rightarrow \mathcal{L}'$, a Θ -invariant $p = (p_P)_{P \in \mathcal{L}'} \in [0, 1]^{\mathcal{L}'}$ with $\sum_{P \in \mathcal{L}'} p_P = 1$ and an open interpretation $I: T \rightsquigarrow T_\Theta$ such that $\phi = \psi_{\Theta, p}^I$.*

3.1 Comparison to ad hoc theories

Hypergraphs. The theory of hypergraphons has been most inspirational to our work as it also pertains to quasirandomness of “different strength”, arranged in hierarchies like ours. In fact, the last three theorems compare our notions with the hierarchies based on various discrepancy properties from the literature.

As we remarked in the introduction, the results of [Tow17] imply that $\text{Dev}[k-1] = \text{Disc}[\mathcal{A}_{k-1}]$ is the strongest discrepancy property below full quasirandomness and $\text{Disc}[\{\ell+1\}]$ is the weakest discrepancy property above $\text{CliqueDisc}[\ell]$. This together with Theorems 3.1, 3.2 and 3.9 and the three theorems below justify the Hasse diagram of Figure 2 between the families **Independence** and **UInduce** and the discrepancy properties in the literature.

The following theorem trivially follows from definitions.

Theorem 3.14. *For every $k \geq \ell \geq 1$ and every $\phi \in \text{Hom}^+(\mathcal{A}[T_{k\text{-Hypergraph}}], \mathbb{R})$, if $\phi \in \text{UInduce}[\ell]$, then $\phi \in \text{CliqueDisc}[\ell]$.*

Theorem 3.15. *For every $k \in \mathbb{N}_+$, there exists $\phi \in \text{Hom}^+(\mathcal{A}[T_{k\text{-Hypergraph}}], \mathbb{R})$ satisfying $\text{Dev}[k-1]$ but not satisfying $\text{UInduce}[1]$.*

Theorem 3.16. *For every $k > \ell \geq 1$, there exists $\phi \in \text{Hom}^+(\mathcal{A}[T_{k\text{-Hypergraph}}], \mathbb{R})$ satisfying $\text{Independence}[\ell]$ but not satisfying $\text{Disc}[\{\ell+1\}]$.*

Table 1 contains pointers to where each of the theorems (or their parts) are proved.

Permutations. In our language, the quasirandom permuton [Coo04, KP13] is simply $\psi_{\text{lin}} \otimes \psi_{\text{lin}}$ (see [CR20, Example 6]). It does not satisfy even the weakest of our properties $\text{UInduce}[1]$. This can be easily verified by a direct computation, but a more instructive way would be to apply Theorem 3.7 and Theorem 3.10(i) $\equiv$ (vii). Since, on the other hand, $\psi_{\text{lin}} \in \text{UInduce}[1]$, we see that the analogue of Theorem 3.4 is not true for unique inducibility.

Theorem	Proof location
3.1	Section 6
3.2	Section 4
3.3	Section 5
3.4	Section 4
3.5	Section 8
3.6	Section 8
3.7	Section 8
3.8	Section 8
3.9	Section 8
3.10 (i) $\equiv$ (ii) $\equiv$ (iii)	Lemma 7.8
(i) $\equiv$ (iv) $\equiv$ (v)	Lemma 4.4
(iv) $\implies$ (vi)	Lemma 4.7
(vi) $\implies$ (vii)	Lemma 7.9
(vii) $\implies$ (ii)	Lemma 7.10
3.11 (i) $\equiv$ (ii)	Lemma 6.1
(iii) $\implies$ (i)	Lemma 6.3
(i) $\implies$ (iii)	Lemma 6.13
3.12	Section 9
3.13	Section 9
3.14	Trivial (see Definitions 2.6 and 2.8)
3.15	Section 8
3.16	Section 8

Table 1: Proof locations for theorems of Section 3.

These observations suggest an interesting research direction; we will return to it in Section 10.

Words. In our language, quasirandom words defined in [HKPS20] are simply $\psi_{\text{lin}} \otimes \psi_p$ ($p \in (0, 1)$, $\psi_p \in \text{Hom}^+(\mathcal{A}[T_{2\text{-Coloring}}], \mathbb{R})$). This is clearly generalizable to $\psi_p \in \text{Hom}^+(\mathcal{A}[T_{c\text{-Coloring}}], \mathbb{R})$ ($p \in \Pi_c$), corresponding to quasirandom word sequences over the alphabet $[c]$ with given letter frequencies $(p_1, \dots, p_c)$. In this way, one can immediately recover existence and uniqueness of the limits of arbitrary (not necessarily quasirandom) convergent sequences from the general theory in [CR20].

In terms of comparisons, since $\psi_p \notin \text{UInduce}[1]$, the same is true for the quasirandom “wordeons” $\psi_{\text{lin}} \otimes \psi_p$.

Latin squares. This is a very interesting example since it is the first time we have encountered an ad hoc theory of limit objects that is provably different from what might be extracted from our framework.

Recall (see e.g. [DK74]) that there are two major forms of representing a Latin square: as a multiplication table of a quasigroup and as an orthogonal array. As it turns out, they lead to different theories.

The limit theory of Latin squares based on the tabular representation was developed in [GHHS20], and the corresponding theory of quasirandomness was continued in [CKLM20]. In the language of theons, this theory can be handled only after a fashion, in the same vein as limits of functions on finite vector spaces [CR20, Sect. 7.5], that is by introducing countably many auxiliary predicate symbols. In this way one immediately gets existence and uniqueness, but other than that the result will be somewhat ugly and not particularly instructive.

The orthogonal array representation opens up another possibility. Recall that in this representation a Latin square is simply an n^2 -subset of $[n] \times [n] \times [n]$ such that its projection onto every two coordinates is bijective. Uniformly sampling from this set, we will get a model of $T_{\text{LinOrder}} \cup T_{\text{LinOrder}} \cup T_{\text{LinOrder}}$. Hence a “Borromean” (as in “Borromean rings”) view of limits of Latin squares would be simply an element of $\text{Hom}^+(\mathcal{A}[T_{\text{LinOrder}} \cup T_{\text{LinOrder}} \cup T_{\text{LinOrder}}], \mathbb{R})$ such that all three permutons obtained from it by erasing one of the orders are quasirandom.

One obvious example is the quasirandom limit of Latin squares $\psi_{\text{lin}} \otimes \psi_{\text{lin}} \otimes \psi_{\text{lin}}$. But there are others. Indeed, in complete analogy with permutons, limits of Latin squares (in our sense) can be uniquely identified with probability distributions on $[0, 1]^3$ such that all three 2-dimensional marginals

are uniform. Under this identification, $\psi_{\text{lin}} \otimes \psi_{\text{lin}} \otimes \psi_{\text{lin}}$ corresponds to the uniform probability measure on $[0, 1]^3$ and a non-quasirandom example is provided, say, by the uniform probability measure supported on the skewed quasi-random graphon (2) with $p = 1/2$.

Finally, since the quasirandom permuton does not satisfy $\text{UInduce}[1]$, it follows that no limit of Latin squares satisfies $\text{UInduce}[1]$ as well.

4 Basic properties and the first equivalence

In this section we present some initial properties about the notions we have defined and prove the easiest equivalence in Theorem 3.10 between items (i), (iv) and (v). The first proposition says that only trivial objects can have unique coupleability parameter greater or equal to its rank; this stems from the fact that non-trivial objects are not uniquely coupleable with themselves.

Proposition 4.1. *Let $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ and $r \stackrel{\text{def}}{=} \text{rk}(\phi)$.*

i. $r = 0$ if and only if $\phi \in \bigcap_{\ell \in \mathbb{N}} \text{UCouple}[\ell]$.

ii. If $r > 0$ then $\phi \notin \text{UCouple}[r]$.

Proof. Note that $r = 0$ if and only if all peons $\mathcal{N}_P$ are trivial (that is, $\mathcal{N}_P = \emptyset$ or $\mathcal{N}_P = \mathcal{E}_{k(P)}$ a.e.), which in turn is equivalent to having $\phi(\langle K \rangle) \in \{0, 1\}$ for every finite set V and every $K \in \mathcal{K}_V[T]$. This implies that there is a unique $K \in \mathcal{K}_V[T]$ with $\phi(\langle K \rangle) = 1$ and this K must further have full automorphism group $\text{Aut}(K) = S_V$.

Let now $\psi \in \text{Hom}^+(\mathcal{A}[T'], \mathbb{R})$ for some theory T' , and assume that ξ is a coupling of ϕ and ψ . Fix a $(T \cup T')$ -on $\mathcal{N}$ such that $\xi = \phi_{\mathcal{N}}$. Then for every $K \in \mathcal{K}_V[T \cup T']$ with V finite we have $T_{\text{ind}}(K, \mathcal{N}) = T_{\text{ind}}(I(K), I(\mathcal{N})) \cap T_{\text{ind}}(I'(K), I'(\mathcal{N}))$, where $I: T \rightsquigarrow T \cup T'$ and $I': T' \rightsquigarrow T \cup T'$ are the structure-erasing interpretations.

If $r = 0$, we get $\xi(\langle K \rangle) = \phi(\langle I(K) \rangle) \psi(\langle I'(K) \rangle)$ (since ϕ is 0-1 valued) so the forward direction of item (i) follows.

The backward direction of item (i) clearly follows from item (ii), so let us prove the latter by contradiction. Suppose that $\phi \in \text{UCouple}[r]$ and fix a T -on $\mathcal{N}$ such that $\phi = \phi_{\mathcal{N}}$ and $\text{rk}(\mathcal{N}) = r$. Consider the $(T \cup T)$ -on $\mathcal{H} \stackrel{\text{def}}{=} \mathcal{N} \dot{\cup} \mathcal{N}$ in which both copies of each predicate symbol P get mapped to $\mathcal{N}_P$, i.e., $\mathcal{H}$ is the coupling of $\mathcal{N}$ with itself. Since $\text{rk}(\mathcal{H}) = \text{rk}(\mathcal{N}) = r$ and $\phi \in \text{UCouple}[r]$, we must have $\phi_{\mathcal{H}} = \phi \otimes \phi$.

Fix a finite set V and $K \in \mathcal{K}_V[T]$ and let $K_2 \in \mathcal{K}_V[T \cup T]$ be given by setting $R_P(K_2) \stackrel{\text{def}}{=} R_P(K)$ for both copies of each predicate symbol P . Then we have

$$\phi(\langle K \rangle) = t_{\text{ind}}(K, \mathcal{N}) = t_{\text{ind}}(K_2, \mathcal{H}) = (\phi \otimes \phi)(\langle K_2 \rangle) = \phi(\langle K \rangle)^2,$$

so we must have $\phi(\langle K \rangle) \in \{0, 1\}$. Hence $r = 0$, and item (ii) follows. $\square$

The next two propositions will make use of the theon uniqueness theorems [CR20, Theorems 3.9 and 3.11, Proposition 7.7]. Recall from [CR20, Definition 3.8 and Sect. 7] that for a sequence of symmetric (i.e., S_d -invariant) functions $f = (f_d)_{d=1}^k$ with $f_d: \mathcal{E}_d(\Omega) \rightarrow \Omega'$ the sequence of functions $\hat{f} = (\hat{f}_d)_{d=1}^k$ with $\hat{f}_d: \mathcal{E}_d(\Omega) \rightarrow \mathcal{E}_d(\Omega')$ is defined by

$$\hat{f}_d(x)_A \stackrel{\text{def}}{=} f_{|A|}(\iota_A^*(x)) \quad (A \in r(d)).$$

As we have seen in the introduction, a positive homomorphism $\phi \in \text{Independence}[\ell]$ can have geometric realizations far from being ℓ -independent (cf. (1) and (2)). The next proposition says that for rank the situation is precisely the opposite.

Proposition 4.2. *For every peon $\mathcal{N} \subseteq \mathcal{E}_k(\Omega)$ there exists another peon $\mathcal{H} \subseteq \mathcal{E}_k(\Omega)$ such that $\text{rk}(\mathcal{H}) = \text{rk}(\phi_{\mathcal{N}})$ and $\mathcal{H} = \mathcal{N}$ a.e. Moreover, if $\mathcal{N}$ is ℓ -independent for some $\ell \leq k$, then $\mathcal{H}$ can be taken to also be ℓ -independent.*

Proof. Let μ be the measure of Ω and X be its underlying space, let $r \stackrel{\text{def}}{=} \text{rk}(\phi_{\mathcal{N}})$ and define the function $W: \mathcal{E}_{k,r}(\Omega) \rightarrow [0, 1]$ by

$$W(x) \stackrel{\text{def}}{=} \mu(\{y \in X^{(\frac{[k]}{>r})} \mid (x, y) \in \mathcal{N}\}), \quad (12)$$

defining it arbitrarily when this set is not measurable. Fubini's Theorem ensures that this function is measurable so we define

$$\mathcal{H} \stackrel{\text{def}}{=} W^{-1}(1) \times X^{(\frac{[k]}{>r})}.$$

Clearly $\text{rk}(\mathcal{H}) \leq r$. Hence, to prove that $\mathcal{H} = \mathcal{N}$ a.e., it is sufficient to show that W is 0-1 valued a.e.

Since $\text{rk}(\phi_{\mathcal{N}}) = r$, we know that there exists a peon $\mathcal{G}$ over some space $\Omega' = (X', \mathcal{A}', \mu')$ such that $\phi_{\mathcal{G}} = \phi_{\mathcal{N}}$ and $\text{rk}(\mathcal{G}) = r$. By theon uniqueness [CR20, Proposition 7.7], there exist sequences $f = (f_d)_{d=1}^k, g = (g_d)_{d=1}^k$ of symmetric

measure preserving on h.o.a. (higher order arguments) functions ($f_d: \mathcal{E}_d \rightarrow \Omega$ and $g_d: \mathcal{E}_d \rightarrow \Omega'$) such that

$$\widehat{f}_k(z) \in \mathcal{N} \equiv \widehat{g}_k(z) \in \mathcal{G} \quad (13)$$

for almost every $z \in \mathcal{E}_k$. From the structure of the function $\widehat{f}_k$, we can decompose it as

$$\widehat{f}_k(x, y) = (F_1(x), F_2(x, y)),$$

for every $(x, y) \in \mathcal{E}_{k,r} \times [0, 1]^{([k]_{>r})}$, where $F_1: \mathcal{E}_{k,r} \rightarrow \mathcal{E}_{k,r}(\Omega)$ and $F_2: \mathcal{E}_k \rightarrow X^{([k]_{>r})}$ are given by

$$F_1(x)_A \stackrel{\text{def}}{=} f_{|A|}(\iota_A^*(x)), \quad F_2(x, y)_A \stackrel{\text{def}}{=} f_{|A|}(\iota_A^*(x, y)).$$

We perform a similar decomposition of $\widehat{g}_k$ in terms of functions $G_1: \mathcal{E}_{k,r} \rightarrow \mathcal{E}_{k,r}(\Omega')$ and $G_2: \mathcal{E}_k \rightarrow (X')^{([k]_{>r})}$.

Since the functions f_d are measure preserving on h.o.a., it follows that F_1 is measure preserving and for every $x \in \mathcal{E}_{k,r}$ the restriction $F_2(x, -): [0, 1]^{([k]_{>r})} \rightarrow X^{([k]_{>r})}$ is measure preserving. Hence Fubini's Theorem applied to (13) implies

$$W(F_1(x)) = \lambda(\{y \in [0, 1]^{([k]_{>r})} \mid (G_1(x), G_2(x, y)) \in \mathcal{G}\})$$

for almost every $x \in \mathcal{E}_{k,r}$. But since $\text{rk}(\mathcal{G}) = r$, the measure above can only be 0 or 1 (as $G_2(x, y)$ contains only coordinates with $|A| > r$). Since F_1 is measure preserving, this implies that $W(z) \in \{0, 1\}$ for almost every $z \in \mathcal{E}_{k,r}(\Omega)$ and thus $\mathcal{H} = \mathcal{N}$ a.e.

We have already shown that $\text{rk}(\mathcal{H}) \leq r$ and since $\mathcal{H} = \mathcal{N}$ μ -a.e. implies $\phi_{\mathcal{H}} = \phi_{\mathcal{N}}$, the other inequality must also hold.

The last statement is obvious from the construction. $\square$

As we have seen in Section 2.1, given an open interpretation $I: T_1 \rightsquigarrow T_2$ and a T_2 -on $\mathcal{H}$, the T_1 -on $I(\mathcal{H})$ represents the limit object constructed from $\phi_{\mathcal{H}}$ via I , i.e., we have $\phi_{I(\mathcal{H})} = \phi_{\mathcal{H}}^I$. However, given a T_1 -on $\mathcal{N}$ and $\phi \in \text{Hom}^+(\mathcal{A}[T_2], \mathbb{R})$ such that $\phi^I = \phi_{\mathcal{N}}$, it is *not* true that there exists a T_2 -on $\mathcal{H}$ such that both $I(\mathcal{H}) = \mathcal{N}$ a.e. and $\phi_{\mathcal{H}} = \phi$ (see [CR20, Example 45]). The next proposition says in essence that this example is the worst that can happen: at the cost of adding an extra dummy variable, we can find an $\mathcal{H}$ such that $I(\mathcal{H})_P = \mathcal{N}_P \times \mathcal{E}_{k(P)}$ a.e. and $\phi_{\mathcal{H}} = \phi$.

Proposition 4.3. *Let $I: T_1 \rightsquigarrow T_2$, let $\phi \in \text{Hom}^+(\mathcal{A}[T_2], \mathbb{R})$, and let $\mathcal{N}$ be a T_1 -on over Ω such that $\phi^I = \phi_{\mathcal{N}}$. Then there exists a T_2 -on $\mathcal{H}$ over $\Omega \times \Omega$ such that $\phi_{\mathcal{H}} = \phi$ and $I(\mathcal{H})_P = \mathcal{N}_P \times \mathcal{E}_{k(P)}(\Omega)$ a.e., for every predicate symbol P in the language of T_1 .*

Furthermore, if $T_2 = T_1 \cup T'$ for some T' and I is the structure-erasing interpretation, then $\mathcal{H}$ can be taken to satisfy $I(\mathcal{H})_P \stackrel{\text{def}}{=} \mathcal{H}_P = \mathcal{N}_P \times \mathcal{E}_{k(P)}(\Omega)$ everywhere for every predicate symbol P in the language of T_1 .

Proof. For $i \in [2]$, let $\mathcal{L}_i$ be the language of T_i and let $k_i \stackrel{\text{def}}{=} \max_{P \in \mathcal{L}_i} k(P)$. Let $\mathcal{G}$ be a T_2 -on over Ω such that $\phi_{\mathcal{G}} = \phi$. Since $\phi_{I(\mathcal{G})} = \phi^I = \phi_{\mathcal{N}}$, by theon uniqueness [CR20, Proposition 7.7], there exists a sequence $h = (h_d)_{d=1}^{k_1}$ of symmetric measure preserving on h.o.a. functions $(h_d: \mathcal{E}_d(\Omega) \times \mathcal{E}_d(\Omega) \rightarrow \Omega)$ such that

$$\widehat{h}_{k(P)}(x, \widehat{x}) \in I(\mathcal{G})_P \equiv x \in \mathcal{N}_P, \quad (14)$$

for every $P \in \mathcal{L}_1$ and almost every $(x, \widehat{x}) \in \mathcal{E}_{k(P)}(\Omega) \times \mathcal{E}_{k(P)}(\Omega)$. Extend the family h by defining $h_d: \mathcal{E}_d(\Omega) \times \mathcal{E}_d(\Omega) \rightarrow \Omega$ for $k_1 < d \leq \max\{k_1, k_2\}$ as $h_d(x, \widehat{x}) \stackrel{\text{def}}{=} x_{[d]}$, and note that h_d is symmetric and measure preserving on h.o.a.

Define then the T_2 -on $\mathcal{H}$ over $\Omega \times \Omega$ by

$$\mathcal{H}_Q \stackrel{\text{def}}{=} \widehat{h}_{k(Q)}^{-1}(\mathcal{G}_Q) \quad (15)$$

for every $Q \in \mathcal{L}_2$. By (the easy direction of) theon uniqueness [CR20, Proposition 7.7], it follows that $\phi_{\mathcal{H}} = \phi_{\mathcal{G}} = \phi$. On the other hand, the definition of $\mathcal{H}$ ensures that

$$(x, \widehat{x}) \in I(\mathcal{H})_P \equiv \widehat{h}_{k(P)}(x, \widehat{x}) \in I(\mathcal{G})_P$$

for every $P \in \mathcal{L}_1$ and every $(x, \widehat{x}) \in \mathcal{E}_{k(P)}(\Omega) \times \mathcal{E}_{k(P)}(\Omega)$, which together with (14) implies $I(\mathcal{H})_P = \mathcal{N}_P \times \mathcal{E}_{k(P)}(\Omega)$ a.e.

For the case when $T_2 = T_1 \cup T'$ for some T' and I is the structure-erasing interpretation, we define $\mathcal{H}$ instead by using (15) only for $Q \in \mathcal{L}_2 \setminus \mathcal{L}_1$ and use $\mathcal{H}_P \stackrel{\text{def}}{=} \mathcal{N}_P \times \mathcal{E}_{k(P)}(\Omega)$ for every $P \in \mathcal{L}_1$ (as required). By (14) this is only a zero-measure change from the previous definition so we still have $\phi_{\mathcal{H}} = \phi$. $\square$

Propositions 4.2 and 4.3 allow us to show the equivalence in Theorem 3.10 between items (i), (iv) and (v).

Lemma 4.4 (Theorem 3.10(i)≡(iv)≡(v)). *The following are equivalent for $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ and $\ell \in \mathbb{N}$.*

- i. $\phi \in \text{UCouple}[\ell]$.
- ii. ϕ is weakly ℓ -independent.
- iii. Every T -on $\mathcal{N}$ with $\phi = \phi_{\mathcal{N}}$ is weakly ℓ -independent.

Proof. (iii) $\implies$ (ii) is trivial.

(ii) $\implies$ (i).

Let $\mathcal{N}$ be a T -on over some space $\Omega = (X, \mathcal{A}, \mu)$ such that the exchangeable array $\mathbf{K}$ corresponding to $\mathcal{N}$ with respect to $\boldsymbol{\theta}$ picked in $\mathcal{E}_{\mathbb{N}_+}(\Omega)$ according to μ is independent from $(\boldsymbol{\theta}_A \mid A \in r(\mathbb{N}_+, \ell))$. Let $\psi \in \text{Hom}^+(\mathcal{A}[T'], \mathbb{R})$ for some theory T' be such that $\text{rk}(\psi) \leq \ell$ and let $\xi \in \text{Hom}^+(\mathcal{A}[T \cup T'], \mathbb{R})$ be any coupling of ϕ and ψ . We have to prove that $\xi = \phi \otimes \psi$.

Let also $I: T \rightsquigarrow T \cup T'$ and $I': T' \rightsquigarrow T \cup T'$ be the structure-erasing interpretations. By Proposition 4.3, there exists a $(T \cup T')$ -on $\mathcal{H}$ over $\Omega \times \Omega$ such that $\xi = \phi_{\mathcal{H}}$ and

$$\mathcal{H}_P = \mathcal{N}_P \times \mathcal{E}_{k(P)}(\Omega) \quad (16)$$

for every P in the language of T . By possibly changing zero-measure sets of the peons corresponding to T' using Proposition 4.2, we may also assume $\text{rk}(I'(\mathcal{H})) = \text{rk}(\psi) \leq \ell$.

Let us pick $\boldsymbol{\eta}$ in $\mathcal{E}_{\mathbb{N}_+}(\Omega)$ according to μ and independently from $\boldsymbol{\theta}$; we view $(\boldsymbol{\theta}, \boldsymbol{\eta})$ as a $\mathcal{E}_{\mathbb{N}_+}(\Omega \times \Omega)$ -valued random variable distributed according to $\mu \otimes \mu$. Let $\mathbf{L}$ be the exchangeable array corresponding to $\mathcal{H}$ with respect to $(\boldsymbol{\theta}, \boldsymbol{\eta})$. Note that (16) implies that $I(\mathbf{L}) = \mathbf{K}$, which in turn implies that $I(\mathbf{L})$ is independent from $((\boldsymbol{\theta}_A \mid A \in r(\mathbb{N}_+, \ell)), \boldsymbol{\eta})$. On the other hand, since $\text{rk}(I'(\mathcal{H})) \leq \ell$, it follows that $I'(\mathbf{L})$ is completely determined by $((\boldsymbol{\theta}_A, \boldsymbol{\eta}_A) \mid A \in r(\mathbb{N}_+, \ell))$, so $I(\mathbf{L})$ is independent from $I'(\mathbf{L})$. This means that for $m \in \mathbb{N}_+$ and $K \in \mathcal{K}_m[T \cup T']$, we have

$$\begin{aligned} \xi(\langle K \rangle) &= \mathbb{P}[\mathbf{L}|_{[m]} = K] = \mathbb{P}[I(\mathbf{L})|_{[m]} = I(K) \wedge I'(\mathbf{L})|_{[m]} = I'(K)] \\ &= \mathbb{P}[I(\mathbf{L})|_{[m]} = I(K)] \cdot \mathbb{P}[I'(\mathbf{L})|_{[m]} = I'(K)] = \phi(\langle I(K) \rangle) \cdot \psi(\langle I'(K) \rangle), \end{aligned}$$

so $\xi = \phi \otimes \psi$, hence item (i) follows.

Let us prove (i) $\implies$ (iii). Let $\Omega = (X, \mathcal{A}, \mu)$ be an atomless complete probability space and $\mathcal{N}$ be a T -on over Ω with $\phi = \phi_{\mathcal{N}}$. We have to prove

that the exchangeable array $\mathbf{K}$ corresponding to $\mathcal{N}$ with respect to $\boldsymbol{\theta}$ picked in $\mathcal{E}_{\mathbb{N}_+}(\Omega)$ according to μ is independent from $(\boldsymbol{\theta}_A \mid A \in r(\mathbb{N}_+, \ell))$. For that, it is sufficient to show that for any $m \in \mathbb{N}$, any $K \in \mathcal{K}_m[T]$ and any measurable set $B \subseteq \mathcal{E}_{m,\ell}(\Omega)$, the events $\mathbf{K}|_{[m]} = K$ and $(\boldsymbol{\theta}_A \mid A \in r(m, \ell)) \in B$ are independent.

Let Q be a new m -ary predicate symbol and consider the $(T \cup T_{\{Q\}})$ -on $\mathcal{H}$ over Ω given by $\mathcal{H}_P \stackrel{\text{def}}{=} \mathcal{N}_P$ for every P in the language of T and $\mathcal{H}_Q \stackrel{\text{def}}{=} B \times X^{(\binom{[m]}{>\ell})}$. Let also $I: T \rightsquigarrow T \cup T_{\{Q\}}$ and $I': T_{\{Q\}} \rightsquigarrow T \cup T_{\{Q\}}$ be the structure-erasing interpretations so that $\phi_{\mathcal{H}}$ is a coupling of ϕ and $\phi'_{\mathcal{H}}$. Since $\text{rk}(\phi'_{\mathcal{H}}) \leq \text{rk}(\mathcal{H}_Q) \leq \ell$ and $\phi \in \text{UCouple}[\ell]$, we have $\phi_{\mathcal{H}} = \phi \otimes \phi'_{\mathcal{H}}$. Finally, let S be the set of all $L \in \mathcal{K}_m[T \cup T_{\{Q\}}]$ such that $I(L) = K$ and $(1, 2, \dots, m) \in R_Q(L)$. Then we have

$$\begin{aligned} \mathbb{P}[\mathbf{K}|_{[m]} = K \wedge (\boldsymbol{\theta}_A \mid A \in r(m, \ell)) \in B] &= \sum_{L \in S} \phi_{\mathcal{H}}(\langle L \rangle) \\ &= \phi(\langle K \rangle) \sum_{L \in S} \phi'_{\mathcal{H}}(\langle I'(L) \rangle) = \mathbb{P}[\mathbf{K}|_{[m]} = K] \cdot \mathbb{P}[(\boldsymbol{\theta}_A \mid A \in r(m, \ell)) \in B], \end{aligned}$$

which completes the proof. $\square$

The alternative characterization of UCouple via weak independence gives easy proofs of Theorems 3.2 and 3.4.

Proof of Theorem 3.2. $\text{Independence}[\ell] \implies \text{UCouple}[\ell]$.

Let $\mathcal{N}$ be an ℓ -independent T -on, and let $\mathbf{K}$ be the exchangeable array corresponding to $\mathcal{N}$. Then each $R_P(\mathbf{K})$ depends only on the coordinates $\boldsymbol{\theta}_A$ with $|A| > \ell$ (see (7)) and hence is independent from $(\boldsymbol{\theta}_A \mid A \in r(A, \ell))$. Therefore, $\mathcal{N}$ is weakly ℓ -independent and $\text{Independence}[\ell] \implies \text{UCouple}[\ell]$ follows from Lemma 4.4.

The implication $\text{UCouple}[\ell] \implies \text{UInduce}[\ell]$ follows trivially from the definitions. $\square$

Proof of Theorem 3.4. For item (i), if $\mathcal{N}^1$ and $\mathcal{N}^2$ are ℓ -independent theons then $\mathcal{N}^1 \otimes \mathcal{N}^2$ is also ℓ -independent, from which the statement follows.

For item (ii), pick arbitrarily theons $\mathcal{N}^1$ and $\mathcal{N}^2$ such that $\phi_i = \phi_{\mathcal{N}^i}$. Let $(\boldsymbol{\theta}^1, \boldsymbol{\theta}^2)$ be uniformly distributed in $\mathcal{E}_{\mathbb{N}_+} \times \mathcal{E}_{\mathbb{N}_+}$, and let $\mathbf{K}$ be the exchangeable array corresponding to $\mathcal{N}^1 \otimes \mathcal{N}^2$ with respect to $(\boldsymbol{\theta}^1, \boldsymbol{\theta}^2)$. Note that for $i \in [2]$ and for the structure-erasing interpretation $I_i: T_i \rightsquigarrow T_1 \cup T_2$, the exchangeable array corresponding to $\mathcal{N}^i$ with respect $\boldsymbol{\theta}^i$ is $I_i(\mathbf{K})$.

By Lemma 4.4, it is sufficient to show that if $I_i(\mathbf{K})$ is independent from $(\boldsymbol{\theta}_A^i \mid A \in r(\mathbb{N}_+, \ell))$ for $i \in [2]$, then $\mathbf{K}$ is independent from $((\boldsymbol{\theta}_A^1, \boldsymbol{\theta}_A^2) \mid A \in r(\mathbb{N}_+, \ell))$. This immediately follows from the following easily verifiable general fact:

Claim 4.5. *Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}_1, \mathbf{Y}_2$ be mutually independent random variables, and let $f_1(X_1, Y_1), f_2(X_2, Y_2)$ be functions such that $f_i(\mathbf{X}_i, \mathbf{Y}_i)$ is independent from $\mathbf{X}_i$ ($i = 1, 2$). Then $(f_1(\mathbf{X}_1, \mathbf{Y}_1), f_2(\mathbf{X}_2, \mathbf{Y}_2))$ is independent from $(\mathbf{X}_1, \mathbf{X}_2)$.*

In our context, we set $\mathbf{X}_i = (\boldsymbol{\theta}_A^i \mid |A| \leq \ell)$, $\mathbf{Y}_i = (\boldsymbol{\theta}_A^i \mid |A| > \ell)$ and let f_i compute the array $I_i(K)$ from (X_i, Y_i) (thus $(f_1(X_1, Y_1), f_2(X_2, Y_2))$ computes the array K from (X_1, X_2, Y_1, Y_2)). $\square$

The next lemma says that unique coupleability satisfies a “chain rule” analogous to the chain rule for mutual independence of random variables.

Lemma 4.6. *Let $\phi_i \in \text{Hom}^+(\mathcal{A}[T_i], \mathbb{R})$ for $i \in [t]$ and suppose that for every $i \in [t-1]$, ϕ_{i+1} is uniquely coupleable with $\phi_1 \otimes \cdots \otimes \phi_i$. Then $\phi_1, \dots, \phi_t$ are uniquely coupleable.*

Proof. The proof is by induction in t . The result for $t = 1$ is trivial. For $t \geq 2$, let $\xi \in \text{Hom}^+(\mathcal{A}[\bigcup_{i=1}^t T_i], \mathbb{R})$ be a coupling of $\phi_1, \dots, \phi_t$ and let $I: \bigcup_{i=1}^{t-1} T_i \rightsquigarrow \bigcup_{i=1}^t T_i$ be the structure-erasing interpretation. Since ξ^I is a coupling of $\phi_1, \dots, \phi_{t-1}$, by inductive hypothesis we must have $\xi^I = \phi_1 \otimes \cdots \otimes \phi_{t-1}$ so ξ is also a coupling of $\phi_1 \otimes \cdots \otimes \phi_{t-1}$ and ϕ_t , hence we must have $\xi = \phi_1 \otimes \cdots \otimes \phi_t$. $\square$

We finish this section with the (almost trivial) implication (iv) $\implies$ (vi) of Theorem 3.10.

Lemma 4.7 (Theorem 3.10(iv) $\implies$ (vi)). *Let $\ell \in \mathbb{N}$. If ϕ is weakly ℓ -independent, then ϕ is ℓ -local.*

Proof. Let $\mathbf{K}$ be the exchangeable array corresponding to some theon $\mathcal{N}$ with respect to $\boldsymbol{\theta}$ picked in $\mathcal{E}_{\mathbb{N}_+}(\Omega)$ according to μ such that $\phi = \phi_{\mathcal{N}}$ and suppose $\mathbf{K}$ is independent from $(\boldsymbol{\theta}_A \mid A \in r(\mathbb{N}_+, \ell))$. Since for $V \in r(\mathbb{N}_+)$ the marginal $\mathbf{K}|_V$ depends only on $(\boldsymbol{\theta}_A \mid A \in r(V))$, the marginals $(\mathbf{K}|_{V_i} \mid i \in I)$ are mutually independent as long as the sets V_i have pairwise intersections of size at most ℓ . This follows from the following general observation.

Claim 4.8. *Let $\mathbf{X}, \mathbf{Y}_1, \dots, \mathbf{Y}_n$ be mutually independent random variables and $f_i(X, Y_i)$ be functions such that $(f_1(\mathbf{X}, \mathbf{Y}_1), \dots, f_n(\mathbf{X}, \mathbf{Y}_n))$ is independent of $\mathbf{X}$. Then $f_1(\mathbf{X}, \mathbf{Y}_1), \dots, f_n(\mathbf{X}, \mathbf{Y}_n)$ are mutually independent.*

In our situation, $\mathbf{X} = (\theta_A \mid A \in r(\mathbb{N}_+, \ell))$, $\mathbf{Y}_i = (\theta_A \mid A \in r(V_i) \setminus r(\mathbb{N}_+, \ell))$ and f_i computes the marginal $K|_{V_i}$ from $(\theta_A \mid A \in r(V_i))$.

This completes the proof that ϕ is ℓ -local. $\square$

5 Naturality

The objective of this section is to show Theorem 3.3, that is, to show that our quasirandomness properties are preserved under open interpretations. For this, we need to do a bit of abstract nonsense.

Recall from [CR20, Sct. 2.2] that the category INT has pushouts (otherwise known as amalgamated sums, fibred coproducts, etc.). More concretely, for open interpretations $I_1: T \rightsquigarrow T_1$ and $I_2: T \rightsquigarrow T_2$, a pushout of (I_1, I_2) is given by the theory T' obtained from $T_1 \cup T_2$ by adding the axioms

$$\forall \vec{x}, (I_1(P)(\vec{x}) \equiv I_2(P)(\vec{x})) \quad (17)$$

for every P in the language of T and the open interpretations $J_i: T_i \rightsquigarrow T'$ ($i \in [2]$) that act identically on the language of T_i so that

$$\begin{array}{ccc} T & \xrightarrow{I_1} & T_1 \\ I_2 \downarrow & & \downarrow J_1 \\ T_2 & \xrightarrow{J_2} & T' \end{array}$$

is commutative and has the standard universality property.

The following theorem says that we can also amalgamate limit objects along pushouts. Let us warn that unless the theory T is trivial (in which case a “canonical” amalgamation is provided by the independent coupling), we are not aware of any natural, functorial construction here.

Theorem 5.1. *Let*

$$\begin{array}{ccc} T & \xrightarrow{I_1} & T_1 \\ I_2 \downarrow & & \downarrow J_1 \\ T_2 & \xrightarrow{J_2} & T' \end{array}$$

be a pushout of INT and let $\phi_1 \in \text{Hom}^+(\mathcal{A}[T_1], \mathbb{R})$ and $\phi_2 \in \text{Hom}^+(\mathcal{A}[T_2], \mathbb{R})$ be such that $\phi_1^{I_1} = \phi_2^{I_2}$. Then there exists $\psi \in \text{Hom}^+(\mathcal{A}[T'], \mathbb{R})$ such that $\psi^{J_1} = \phi_1$ and $\psi^{J_2} = \phi_2$.

Proof. First we claim that it is enough to show the case when T' is obtained from $T_1 \cup T_2$ by adding the axioms (17). Indeed, if ψ is constructed for such particular case, then we can get our desired element of $\text{Hom}^+(\mathcal{A}[T'], \mathbb{R})$ for a general pushout T' as ψ^I for the universal isomorphism I between the pushout theories.

Let us prove then the particular case. Let $\mathcal{L}$, $\mathcal{L}_1$ and $\mathcal{L}_2$ be the languages of T , T_1 and T_2 , respectively. For $i \in [2]$, let $\mathcal{N}^i$ be a T_i -on (over $[0, 1]$) such that $\phi_i = \phi_{\mathcal{N}^i}$. Since $\phi_{I_1(\mathcal{N}^1)} = \phi_1^{I_1} = \phi_2^{I_2} = \phi_{I_2(\mathcal{N}^2)}$, by Proposition 4.3, there exists a T_1 -on $\mathcal{H}^1$ over $[0, 1]^2$ such that $I_1(\mathcal{H}^1)_P = I_2(\mathcal{N}^2)_P \times \mathcal{E}_{k(P)}$ λ -a.e. for every $P \in \mathcal{L}$.

Define then the Euclidean structure $\mathcal{H}$ on $\mathcal{L}_1 \cup \mathcal{L}_2$ over $[0, 1]^2$ by

$$\mathcal{H}_P \stackrel{\text{def}}{=} \begin{cases} \mathcal{H}_P^1, & \text{if } P \in \mathcal{L}_1; \\ \mathcal{N}_P^2 \times \mathcal{E}_{k(P)}, & \text{if } P \in \mathcal{L}_2. \end{cases}$$

Let us show that $\mathcal{H}$ is a (weak) T' -on. To show this, it is enough to show (see [CR20, Definition 3.5, Remark 5, Theorem 3.7], by reaxiomatizing T, T_1, T_2 to be substitutionally closed, T' also becomes substitutionally closed) that $T(I_1(P), \mathcal{H}) = T(I_2(P), \mathcal{H})$ λ -a.e. for every $P \in \mathcal{L}$. But this follows from

$$\begin{aligned} T(I_1(P), \mathcal{H}) &= T(I_1(P), \mathcal{H}^1) = I_1(\mathcal{H}^1)_P; \\ T(I_2(P), \mathcal{H}) &= T(I_2(P), \mathcal{N}^2) \times \mathcal{E}_{k(P)} = I_2(\mathcal{N}^2)_P \times \mathcal{E}_{k(P)}. \end{aligned}$$

Finally, since we trivially have $J_1(\mathcal{H}) = \mathcal{H}^1$ and $J_2(\mathcal{H})_P = \mathcal{N}_P^2 \times \mathcal{E}_{k(P)}$ for every $P \in \mathcal{L}_2$, it follows that $\psi \stackrel{\text{def}}{=} \phi_{\mathcal{H}}$ satisfies $\psi^{J_1} = \phi_1$ and $\psi^{J_2} = \phi_2$. $\square$

The next proposition makes use of this amalgamation property to “lift” couplings through interpretations.

Proposition 5.2 (Coupling lifting). *Let $I: T_1 \rightsquigarrow T_2$ be an open interpretation, let T be a canonical theory and let $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ and $\phi_2 \in \text{Hom}^+(\mathcal{A}[T_2], \mathbb{R})$. If ξ is a coupling of ϕ_2^I and ϕ , then there exists a coupling $\hat{\xi}$ of ϕ_2 and ϕ such that $\xi = \hat{\xi}^{I \cup \text{id}_T}$.*

Proof. This follows from Theorem 5.1 and the fact that

$$\begin{array}{ccc} T_1 & \xrightarrow{I} & T_2 \\ \downarrow & & \downarrow \\ T_1 \cup T & \xrightarrow{I \cup \text{Id}_T} & T_2 \cup T \end{array}$$

is a pushout in INT , where the vertical arrows are the structure-erasing interpretations. $\square$

Equipped with this “lifting” construction, we can prove Theorem 3.3 about naturality of our properties.

Proof of Theorem 3.3. For item (i), let $I_i: T_i \rightsquigarrow T \cup T_i$ be the structure-erasing interpretation for $i \in [2]$ and note that if ξ is a coupling of ϕ^I with ψ , then Proposition 5.2 gives us a coupling $\widehat{\xi}$ of ϕ with $\widehat{\psi}$ such that $\xi = \widehat{\xi}^{I \cup \text{Id}_T}$. Since ϕ is uniquely coupleable with ψ we must have $\widehat{\xi} = \phi \otimes \psi$, from which we get $\xi = \widehat{\xi}^{I \cup \text{Id}_T} = \phi^I \otimes \psi$, hence ϕ^I and ψ are uniquely coupleable.

Item (ii) follows trivially from the fact that if $\mathcal{N}$ is an ℓ -independent T_2 -on with $\phi = \phi_{\mathcal{N}}$, then $I(\mathcal{N})$ is an ℓ -independent T_1 -on with $\phi_{I(\mathcal{N})} = \phi^I$.

Item (iii) follows trivially from item (i).

For item (iv), we let $\psi \in \text{Hom}^+(\mathcal{A}[T_{\ell\text{-Hypergraph}}], \mathbb{R})$ and ξ be a coupling of ϕ^I with ψ and we make the same construction of the coupling $\widehat{\xi}$ of ϕ and ψ of item (i) using Proposition 5.2. For $i \in [2]$, let $J_i: T_{\ell\text{-Hypergraph}} \rightsquigarrow T_i \cup T_{\ell\text{-Hypergraph}}$ be the structure-erasing interpretation and note that if $M \in \mathcal{M}[T_1 \cup T_{\ell\text{-Hypergraph}}]$ is such that $J_1(M) \cong K_{|M|}^{(\ell)}$, then we have

$$\begin{aligned} \xi(M) &= \widehat{\xi}^{I \cup \text{Id}_T}(M) \\ &= \widehat{\xi} \left(\sum \left\{ M' \in \mathcal{M}_{|M|}[T_2 \cup T_{\ell\text{-Hypergraph}}] \mid I(I_2(M')) \cong I_1(M) \wedge J_2(M') \cong K_{|M|}^{(\ell)} \right\} \right) \\ &= \psi(K_{|M|}^{(\ell)}) \cdot \phi \left(\sum \left\{ M' \in \mathcal{M}_{|M|}[T_2] \mid I(M') \cong I_1(M) \right\} \right) \\ &= \psi(K_{|M|}^{(\ell)}) \cdot \phi^I(I_1(M)) \\ &= (\phi^I \otimes \psi)(M), \end{aligned}$$

where the third equality follows from the fact that $\phi \in \text{UInduce}[\ell]$. Hence $\phi^I \in \text{UInduce}[\ell]$. $\square$

6 Unique inducibility

In this section we prove Theorem 3.11. We start by showing the equivalence between items (i) and (ii). Curiously, the case $\ell = 1$ is the hardest one to prove.

Lemma 6.1 (Theorem 3.11(i) $\equiv$ (ii)). *Let $\ell \in \mathbb{N}_+$ and $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$. Then $\phi \in \text{UInduce}[\ell]$ if and only if there exists $p \in (0, 1)$ such that ϕ is uniquely inducible by every $\psi \in \text{Hom}^+(\mathcal{A}[T_\ell\text{-Hypergraph}], \mathbb{R})$ with $\psi(\rho_\ell) = p$.*

Proof. The forward implication is obvious.

For $p \in (0, 1)$, let us say that ϕ is uniquely p -inducible if it is uniquely inducible by every $\psi \in \text{Hom}^+(\mathcal{A}[T_\ell\text{-Hypergraph}], \mathbb{R})$ with $\psi(\rho_\ell) = p$. Then the backward implication amounts to showing that unique p -inducibility implies unique q -inducibility for every $p, q \in (0, 1)$ (the cases $q \in \{0, 1\}$ are trivial).

Let $I: T \rightsquigarrow T \cup T_\ell\text{-Hypergraph}$ and $J: T_\ell\text{-Hypergraph} \rightsquigarrow T \cup T_\ell\text{-Hypergraph}$ be the structure-erasing interpretations. Let us assume that ϕ is uniquely p -inducible and let us show that ϕ is uniquely inducible by any $\psi \in \text{Hom}^+(\mathcal{A}[T_\ell\text{-Hypergraph}], \mathbb{R})$ with $\psi(\rho_\ell) = q$. Let ξ be a coupling of ϕ and ψ .

Our objective is to prove that for every $m \in \mathbb{N}$ and every $M \in \mathcal{M}_m[T \cup T_\ell\text{-Hypergraph}]$ with $J(M) \cong K_m^{(\ell)}$ we have

$$\xi(M) = \phi(I(M))\psi(K_m^{(\ell)}). \quad (18)$$

For $m < \ell$ this is trivial (as $\psi(K_m^{(\ell)}) = 1$), so suppose $m \geq \ell$.

Let $I': T_\ell\text{-Hypergraph} \rightsquigarrow T_\ell\text{-Hypergraph} \cup T_2\text{-Coloring}$ be an open interpretation (to be specified later); note that the diagram

$$\begin{array}{ccccc}
 & & T & & \\
 & \swarrow I & & \searrow I & \\
 T_\ell\text{-Hypergraph} & & T \cup T_\ell\text{-Hypergraph} & & T_\ell\text{-Hypergraph} \\
 & \searrow J & & \swarrow J & \\
 & & T \cup T_\ell\text{-Hypergraph} & & \\
 & & \downarrow \text{id}_T \cup I' & & \\
 & & T \cup T_\ell\text{-Hypergraph} \cup T_2\text{-Coloring} & & \\
 & \swarrow I' & & \searrow I' & \\
 & & T_\ell\text{-Hypergraph} \cup T_2\text{-Coloring} & &
 \end{array} \quad (19)$$

is commutative, where the unlabeled arrows are structure-erasing interpretations. For $t \in [0, 1]$ let $\widehat{\xi}_t \stackrel{\text{def}}{=} \xi \otimes \psi_t$ be the independent coupling of ξ and the 2-coloring ψ_t of densities $(t, 1-t)$ (see Definition 2.12); note that the fact that (19) is commutative implies that $\widehat{\xi}_t^{\text{id}_T \cup I'}$ is a coupling of ϕ and $(\psi \otimes \psi_t)^{I'}$.

We start by showing (18) in the case $p \leq q$. In this case, we take

$$I'(E)(x_1, \dots, x_\ell) \stackrel{\text{def}}{=} E(x_1, \dots, x_\ell) \wedge \bigwedge_{i \in [\ell]} \chi_1(x_i),$$

that is, I' keeps edges that are monochromatic in color 1. Let $t \stackrel{\text{def}}{=} (p/q)^{1/\ell}$ and note that for $n \geq \ell$ we have

$$(\psi \otimes \psi_t)^{I'}(K_n^{(\ell)}) = \psi(K_n^{(\ell)})t^n = \psi(K_n^{(\ell)}) \left(\frac{p}{q}\right)^{n/\ell},$$

which in particular implies that $(\psi \otimes \psi_t)^{I'}(\rho_\ell) = p$. On the other hand, we also have $\widehat{\xi}_t^{\text{id}_T \cup I'}(M) = \xi(M)t^m$, so unique p -inducibility of ϕ gives

$$\xi(M)t^m = \widehat{\xi}_t^{\text{id}_T \cup I'}(M) = \phi(I(M))(\psi \otimes \psi_t)^{I'}(K_m^{(\ell)}) = \phi(I(M))\psi(K_m^{(\ell)})t^m,$$

from which (18) follows.

We now show (18) in the case $\ell \geq 2$ and $q < p$. In this case, we let

$$I'(E)(x_1, \dots, x_\ell) \stackrel{\text{def}}{=} \left(E(x_1, \dots, x_\ell) \wedge \bigwedge_{i \in [\ell]} \chi_1(x_i) \right) \vee \bigwedge_{i \in [\ell]} \chi_2(x_i)$$

that is, I' declares edges to be either old edges that are monochromatic in color 1 or any ℓ -set that is monochromatic in color 2. Let $f(x) \stackrel{\text{def}}{=} x^\ell q + (1-x)^\ell$ and note that $f(0) = 1$ and $f(1) = q$, so there exists $t \in (0, 1)$ such that $f(t) = p$. Since $\ell \geq 2$, for $n \geq \ell$, we have

$$(\psi \otimes \psi_t)^{I'}(K_n^{(\ell)}) = \psi(K_n^{(\ell)})t^n + (1-t)^n,$$

which in particular implies that $(\psi \otimes \psi_t)^{I'}(\rho_\ell) = f(t) = p$. On the other hand, we also have $\widehat{\xi}_t^{\text{id}_T \cup I'}(M) = \xi(M)t^m + \phi(I(M))(1-t)^m$, so unique p -inducibility of ϕ gives

$$\begin{aligned} \xi(M)t^m + \phi(I(M))(1-t)^m &= \widehat{\xi}_t^{\text{id}_T \cup I'}(M) = \phi(I(M))(\psi \otimes \psi_t)^{I'}(K_m^{(\ell)}) \\ &= \phi(I(M))(\psi(K_m^{(\ell)})t^m + (1-t)^m), \end{aligned}$$

from which (18) follows.

The case $q < p$ and $\ell = 1$ is more complicated as the construction analogous to the above does not work: cliques in arity 1 need not be monochromatic.

Let us prove first the sub-case $q = p^2$. The idea, roughly speaking, is that when $\ell = 1$, unique p -inducibility says that any “subset of vertices” of relative size p in ϕ induces ϕ and since a “subset of vertices” of relative size p^2 can be seen as having relative size p in some “subset of vertices” that itself has relative size p in the whole space, it must also induce ϕ .

It is worth noting that this idea can be implemented almost literally in the geometric language. But that would require working with theons that have different ground sets in different coordinates so we prefer to present a syntactic argument instead, similar to the one above.

We work with the theory $T_{2\text{-Coloring}}$ instead of $T_{1\text{-Hypergraph}}$ (see Remark 1). Let ξ be a coupling of ϕ and $\psi \stackrel{\text{def}}{=} \psi_{p^2} \in \text{Hom}^+(\mathcal{A}[T_{2\text{-Coloring}}], \mathbb{R})$; we want to show that for every $M \in \mathcal{M}[T \cup T_{2\text{-Coloring}}]$ with $R_{\chi_1}(M) = V(M)$, we have

$$\xi(M) = \phi(I(M))p^{2m},$$

where $m \stackrel{\text{def}}{=} |M|$ and $I: T \rightsquigarrow T \cup T_{2\text{-Coloring}}$ is the structure-erasing interpretation.

Let $I_1, I_2: T_{2\text{-Coloring}} \rightsquigarrow T_{3\text{-Coloring}}$ be the interpretations given by

$$\begin{aligned} I_1(\chi_1)(x) &\stackrel{\text{def}}{=} \chi_1(x) \vee \chi_2(x); & I_2(\chi_1)(x) &\stackrel{\text{def}}{=} \chi_1(x); \\ I_1(\chi_2)(x) &\stackrel{\text{def}}{=} \chi_3(x); & I_2(\chi_2)(x) &\stackrel{\text{def}}{=} \chi_2(x) \vee \chi_3(x). \end{aligned}$$

Let $\widehat{\psi} \stackrel{\text{def}}{=} \psi_{(p^2, p-p^2, 1-p)} \in \text{Hom}^+(\mathcal{A}[T_{3\text{-Coloring}}], \mathbb{R})$ and note that $\widehat{\psi}^{I_i} = \psi_{p^i}$ for $i \in [2]$.

Let $J: T_{2\text{-Coloring}} \rightsquigarrow T \cup T_{2\text{-Coloring}}$ and $\widehat{J}: T_{3\text{-Coloring}} \rightsquigarrow T \cup T_{3\text{-Coloring}}$ be the structure-erasing interpretations. Our definitions ensure that the following

diagram is commutative.

$$\begin{array}{ccccc}
& & T & & \\
& \swarrow I & & \searrow I & \\
T_2\text{-Coloring} & & & & T_2\text{-Coloring} \\
& \searrow J & & \swarrow J & \\
& T \cup T_2\text{-Coloring} & & T \cup T_2\text{-Coloring} & \\
& \searrow \text{id}_T \cup I_1 & & \swarrow \text{id}_T \cup I_2 & \\
& & T \cup T_3\text{-Coloring} & & \\
& \nearrow \hat{J} & & \nwarrow & \\
& & T_3\text{-Coloring} & &
\end{array}
\tag{20}$$

For every $n \in \mathbb{N}$, let $C_n \in \mathcal{M}_n[T_2\text{-Coloring}]$ be the unique model with all vertices satisfying χ_1 .

Since $\hat{\psi}^{I_2} = \psi$, by Proposition 5.2, there exists a coupling $\hat{\xi}$ of ϕ and $\hat{\psi}$ such that $\hat{\xi}^{\text{id}_T \cup I_2} = \xi$. We now make use of the operator $\pi^{(\neg\chi_3, \text{id}_T \cup I_2)}: \mathcal{A}[T \cup T_2\text{-Coloring}] \rightarrow \mathcal{A}_u[T \cup T_3\text{-Coloring}]$ [Raz07, Definition 4], where $u = \sum\{N \in \mathcal{M}_1[T \cup T_3\text{-Coloring}] \mid I_1(\hat{J}(N)) \cong C_1\}$ and $\mathcal{A}_u[T \cup T_3\text{-Coloring}]$ is the localization by the multiplicative system $\{u, u^2, \dots, u^n, \dots\}$. Intuitively, it corresponds to applying the interpretation $\text{id}_T \cup I_2$, followed by throwing away vertices of color 3. (All densities have to be re-normalized by a power of u , this is why we need to localize.) Since

$$\hat{\xi}(u) = \hat{\xi}^{\hat{J} \circ I_1}(C_1) = \hat{\psi}^{I_1}(C_1) = p > 0, \tag{21}$$

we can apply [Raz07, Theorem 2.6] and form the element $\zeta \stackrel{\text{def}}{=} \hat{\xi} \circ \pi^{(\neg\chi_3, \text{id}_T \cup I_2)} \in \text{Hom}^+(\mathcal{A}[T \cup T_2\text{-Coloring}], \mathbb{R})$. We claim that $\zeta^I = \phi$.

To see this, note that for $N \in \mathcal{M}[T]$, we have

$$\zeta^I(N) = \frac{\sum_{N'} \hat{\xi}(N')}{\hat{\xi}(u)^{|N|}},$$

where the sum is over all $N' \in \mathcal{M}_{|N|}[T \cup T_3\text{-Coloring}]$ such that $I((\text{id}_T \cup I_2)(N')) \cong N$ and $J((\text{id}_T \cup I_1)(N')) \cong C_{|N|}$. But since (20) is commutative, the condition $I((\text{id}_T \cup I_2)(N')) \cong N$ is equivalent to $I((\text{id}_T \cup I_1)(N')) \cong N$, which together with (21) gives

$$\zeta^I(N) = \frac{\hat{\xi}^{\text{id}_T \cup I_1}(\hat{N})}{p^{|N|}},$$

where $\widehat{N} \in \mathcal{M}_{|N|}[T \cup T_{2\text{-Coloring}}]$ is the unique model such that $I(\widehat{N}) \cong N$ and $J(\widehat{N}) \cong C_{|N|}$.

Since $\widehat{\xi}^{(\text{id}_T \cup I_1) \circ J}(C_1) = \widehat{\psi}^{I_1}(C_1) = p$ and $\widehat{\xi}^{(\text{id}_T \cup I_1) \circ I} = \xi^I = \phi$, unique p -inducibility of ϕ implies that $\widehat{\xi}^{\text{id}_T \cup I_1}(\widehat{N}) = p^{|N|}\phi(N)$ and thus $\zeta^I = \phi$.

Now we claim that $\zeta^J = \psi_p$. Indeed, note that

$$\begin{aligned} \zeta^J(C_1) &= \frac{\sum \{\widehat{\xi}(N) \mid N \in \mathcal{M}_1[T \cup T_{3\text{-Coloring}}] \wedge J((\text{id}_T \cup I_2)(N)) \cong J((\text{id}_T \cup I_1)(N)) \cong C_1\}}{\widehat{\xi}(u)} \\ &= \frac{\widehat{\xi}^J(\widehat{C}_1)}{p} = \frac{\widehat{\psi}(\widehat{C}_1)}{p} = p, \end{aligned}$$

where $\widehat{C}_1 \in \mathcal{M}_1[T_{3\text{-Coloring}}]$ is the model whose unique vertex satisfies χ_1 , hence $\zeta^J = \psi_p$.

This means that ζ is a coupling of ϕ and ψ_p , so for our fixed $M \in \mathcal{M}_m[T \cup T_{2\text{-Coloring}}]$ with $R_{\chi_1}(M) = V(M)$, unique p -inducibility of ϕ gives

$$\begin{aligned} \xi(M) &= \widehat{\xi}^{\text{id}_T \cup I_2}(M) = \widehat{\xi}(\pi^{(\neg\chi_3, \text{id}_T \cup I_2)}(M)) \cdot \widehat{\xi}(u)^m \\ &= \zeta(M) \cdot p^m = \phi(I(M)) \cdot p^{2m}, \end{aligned}$$

as desired.

From the case $\ell = 1$ and $q = p^2 < p$, with a simple induction, we can derive the case when $\ell = 1$ and $q = p^{2^k} < p$ for some $k \in \mathbb{N}_+$.

Finally, for the case $\ell = 1$ and arbitrary $q < p$, we let $k \in \mathbb{N}_+$ be large enough so that $p^{2^k} < q$ and putting together the previous cases gives that unique p -inducibility implies unique p^{2^k} -inducibility, which in turn implies unique q -inducibility. $\square$

The rest of this section is devoted to various relations between the unique inducibility and the clique discrepancy for hypergraphons; we will also use our findings to prove the last remaining equivalence (i) $\equiv$ (iii) in Theorem 3.11.

It was proved in [Tow17, AHCH⁺18] that for $\ell < k$, $\text{CliqueDisc}[\ell]$ is equivalent to the non-induced labeled density of every ℓ -linear hypergraph H (i.e., hypergraphs whose edges have pairwise intersections of size at most ℓ) being $p^{e(H)}$. We restate below this result in the language of exchangeable arrays.

Theorem 6.2 ([Tow17, AHCH⁺18]). *Let $\ell \in [k-1]$, let $\phi \in \text{Hom}^+(\mathcal{A}[T_{k\text{-Hypergraph}}], \mathbb{R})$ and let $\mathbf{K}$ be the corresponding exchangeable array. Then $\phi \in \text{CliqueDisc}[\ell]$ if and only if for every finite collection $(V_i)_{i \in I}$ of finite subsets of $\mathbb{N}_+$ of size k each and with pairwise intersections of size at most ℓ we have*

$$\mathbb{P}[\forall i \in I, \mathbf{K}|_{V_i} \cong \rho_k] = \prod_{i \in I} \mathbb{P}[\mathbf{K}|_{V_i} \cong \rho_k].$$

Even though this theorem only makes sense in the theory of hypergraphs, we can derive the implication (iii) $\implies$ (i) of Theorem 3.11 for general theories from it.

Lemma 6.3 (Theorem 3.11(iii) $\implies$ (i)). *If $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ is symmetrically ℓ -local, then $\phi \in \text{UInduce}[\ell]$.*

Proof. Let $I: T \rightsquigarrow T \cup T_{\ell\text{-Hypergraph}}$ and $J: T_{\ell\text{-Hypergraph}} \rightsquigarrow T \cup T_{\ell\text{-Hypergraph}}$ be the structure-erasing interpretations.

Our objective is to show that for every $\psi \in \text{Hom}^+(\mathcal{A}[T_{\ell\text{-Hypergraph}}], \mathbb{R})$, every coupling ξ of ϕ and ψ , every $m \in \mathbb{N}$ and every $M \in \mathcal{M}_m[T \cup T_{\ell\text{-Hypergraph}}]$ with $J(M) \cong K_m^{(\ell)}$, we have

$$\xi(M) = \phi(I(M))\psi(K_m^{(\ell)}). \quad (22)$$

Let us first consider the case $m \leq \ell$. In this case, note that for the exchangeable array $\mathbf{K}$ corresponding to ϕ , by letting $V_1 = V_2 = [m]$, symmetric ℓ -locality of ϕ gives

$$\phi(I(M)) = \mathbb{P}[\mathbf{K}|_{[m]} \cong I(M)] = \mathbb{P}[\mathbf{K}|_{[m]} \cong I(M)]^2 = \phi(I(M))^2,$$

so $\phi(I(M)) \in \{0, 1\}$, hence (22) follows.

Suppose now that $m > \ell$ and let $I': T_{m\text{-Hypergraph}} \rightsquigarrow T$ be the open interpretation that declares m -edges to be *isomorphic* copies of $I(M)$, that is, it is given by

$$I'(E)(x_1, \dots, x_m) \stackrel{\text{def}}{=} \bigvee_{\sigma \in S_m} D_{\text{open}}(I(M))(x_{\sigma(1)}, \dots, x_{\sigma(m)}).$$

Let us show that $\phi^{I'} \in \text{Hom}^+(\mathcal{A}[T_{m\text{-Hypergraph}}], \mathbb{R})$ satisfies $\text{CliqueDisc}[\ell]$. Let $\mathbf{K}$ be the exchangeable array corresponding to ϕ so that $I'(\mathbf{K})$ is the exchangeable array corresponding to $\phi^{I'}$. Then if $(V_i)_{i \in [t]}$ is a finite collection

of finite subsets of $\mathbb{N}_+$ of size m each and with pairwise intersections of size at most ℓ , then

$$\begin{aligned}\mathbb{P}[\forall i \in [t], I'(\mathbf{K})|_{V_i} \cong \rho_m] &= \mathbb{P}[\forall i \in [t], \mathbf{K}|_{V_i} \cong M] \\ &= \prod_{i \in [t]} \mathbb{P}[\mathbf{K}|_{V_i} \cong M] = \prod_{i \in [t]} \mathbb{P}[I'(\mathbf{K})|_{V_i} \cong \rho_m],\end{aligned}$$

where the second equality follows from the fact that ϕ is symmetrically ℓ -local. By Theorem 6.2, it follows that $\phi^{I'}$ satisfies $\mathbf{CliqueDisc}[\ell]$.

Note now that the diagram

$$\begin{array}{ccccc} T_m\text{-Hypergraph} & \longrightarrow & T_m\text{-Hypergraph} \cup T_\ell\text{-Hypergraph} & \longleftarrow & T_\ell\text{-Hypergraph} \\ I' \downarrow & & I' \cup \text{id}_{T_\ell\text{-Hypergraph}} \downarrow & & \swarrow J \\ T & \xrightarrow{I} & T \cup T_\ell\text{-Hypergraph} & & \end{array}$$

is commutative, where the unlabeled arrows are structure-erasing interpretations. This implies that $\xi^{I' \cup \text{id}_{T_\ell\text{-Hypergraph}}}$ is a coupling of $\phi^{I'}$ and ψ , so we get

$$\xi(M) = \xi^{I' \cup \text{id}_{T_\ell\text{-Hypergraph}}}(K_m^{(m,\ell)}) = \phi^{I'}(\rho_m)\psi(K_m^{(\ell)}) = \phi(I(M))\psi(K_m^{(\ell)}),$$

where the second equality follows from $\phi^{I'} \in \mathbf{CliqueDisc}[\ell]$. $\square$

Let us now prove an important fact about $\mathbf{CliqueDisc}[\ell]$ and ℓ -flattenings defined below.

Definition 6.4. For a peon $\mathcal{N}$ over $\Omega = (X, \mathcal{A}, \mu)$ and $\ell \in \mathbb{N}$, the ℓ -flattening of $\mathcal{N}$ is the function $W_{\mathcal{N}}^\ell: \mathcal{E}_{k,\ell}(\Omega) \rightarrow [0, 1]$ defined by

$$W_{\mathcal{N}}^\ell(x) \stackrel{\text{def}}{=} \mu(\{y \in X^{(\geq \ell)} \mid (x, y) \in \mathcal{N}\}),$$

and defined arbitrarily when the set above is not measurable.

Note that the construction in (12) is precisely an ℓ -flattening, and so is the construction of a graphon in the ordinary sense from $T_{\text{Graph-on}}$ (cf. (1), (2) and (5)).

Lemma 6.5. *Let $\mathcal{N}$ be a $T_k\text{-Hypergraph-on}$ over $\Omega = (X, \mathcal{A}, \mu)$ such that $\phi_{\mathcal{N}}$ satisfies $\mathbf{CliqueDisc}[\ell]$. Then $W_{\mathcal{N}}^\ell = \phi_{\mathcal{N}}(\rho_k)$ a.e.*

Proof. It is sufficient to prove that the two measures on $X^{r(k,\ell)}$ given by $Y \mapsto \int_Y W_{\mathcal{N}}^\ell d\mu$ and $\nu(Y) \stackrel{\text{def}}{=} \phi_{\mathcal{N}}(\rho_k)\mu(Y)$ coincide, and for that we only have to consider the basis of our σ -algebra, i.e., sets of the form

$$Y = \prod_{A \in r(k,\ell)} V_A.$$

In other words, for every collection $V_A \subseteq X$ ($A \in r(k,\ell)$) of measurable sets we have to prove that

$$\int_Y W_{\mathcal{N}}^\ell d\mu = \phi_{\mathcal{N}}(\rho_k) \cdot \mu(Y). \quad (23)$$

Recall from [Tow17, AHCH⁺18] that **CliqueDisc** $[\ell]$ is equivalent to **Disc** $[\binom{[k]}{\ell}]$ (see Definition 2.8) and for the language $\mathcal{L}_{\binom{[k]}{\ell}}$ containing one predicate symbol P_A of arity ℓ for each $A \in \binom{[k]}{\ell}$, define the $T_{\mathcal{L}_{\binom{[k]}{\ell}}} \cup T_{k\text{-Hypergraph-on } \mathcal{H}}$ over Ω by

$$\mathcal{H}_E \stackrel{\text{def}}{=} \mathcal{N}_E; \quad \mathcal{H}_{P_A} \stackrel{\text{def}}{=} \iota_A^*(Y) = \{x \in \mathcal{E}_\ell(\Omega) \mid \forall A' \in r(A), x_{\iota_A^{-1}(A')} \in V_{A'}\}.$$

Let then $\mathbf{K}$ be the exchangeable array corresponding to $\mathcal{H}$. Since $\phi_{\mathcal{N}}$ satisfies **CliqueDisc** $[\ell] = \text{Disc}[\binom{[k]}{\ell}]$, we get

$$\begin{aligned} \int_Y W_{\mathcal{N}}^\ell d\mu &= \mathbb{P} \left[(1, \dots, k) \in R_E(\mathbf{K}) \wedge \forall A \in \binom{[k]}{\ell}, \iota_A \in R_{P_A}(\mathbf{K}) \right] \\ &= \phi_{\mathcal{N}}(\rho_k) \cdot \mathbb{P} \left[\forall A \in \binom{[k]}{\ell}, \iota_A \in R_{P_A}(\mathbf{K}) \right] \\ &= \phi_{\mathcal{N}}(\rho_k) \cdot \mu(Y), \end{aligned}$$

as desired. $\square$

To prove the final implication (i) $\implies$ (iii) in Theorem 3.11, we will need a small generalization of the easier direction of Theorem 6.2 for disjoint unions of theories of hypergraphs.

Definition 6.6 ($\vec{k}$ -hypergraphs). Given $\vec{k} = (k_1, \dots, k_t) \in \mathbb{N}_+^t$, we let $T_{\vec{k}\text{-Hypergraph}} \stackrel{\text{def}}{=} \bigcup_{i \in [t]} T_{k_i\text{-Hypergraph}}$ and in this theory, we denote the predicate symbol corresponding to the i -th hypergraph by E_i . Models of $T_{\vec{k}\text{-Hypergraph}}$ will be called $\vec{k}$ -hypergraphs and for one such model M , we let $E_i(M) \stackrel{\text{def}}{=} \{\text{im}(\alpha) \mid$

$\alpha \in R_{E_i}(M)\}$ be its i -th edge set. We also denote by $I_i: T_{k_i}\text{-Hypergraph} \rightsquigarrow T_{\vec{k}}\text{-Hypergraph}$ the structure-erasing interpretation corresponding to the i -th edge set.

Proposition 6.7. *Let $\vec{k} = (k_1, \dots, k_t)$, let $\ell \leq \min_{i \in [t]} k_i$, let $i_1, \dots, i_s \in [t]$ and let $(V_j)_{j=1}^s$ be such that $V_j \in \binom{\mathbb{N}_+}{k_{i_j}}$ and $|V_j \cap V_{j'}| \leq \ell$, whenever $j \neq j'$.*

Let $\phi \in \text{Hom}^+(\mathcal{A}[T_{\vec{k}}\text{-Hypergraph}], \mathbb{R})$ be such that all ϕ^{I_i} ($i \in [t]$) satisfy $\text{CliqueDisc}[\ell]$ and let $\mathbf{K}$ be the corresponding exchangeable array. Then

$$\mathbb{P}[\forall j \in [s], V_j \in E_{i_j}(\mathbf{K})] = \prod_{j \in [s]} \mathbb{P}[V_j \in E_{i_j}(\mathbf{K})].$$

Proof. Let $\mathcal{N}$ be a $T_{\vec{k}}\text{-Hypergraph-on}$ such that $\phi_{\mathcal{N}} = \phi$ and note that

$$\begin{aligned} \mathbb{P}[\forall j \in [s], V_j \in E_{i_j}(\mathbf{K})] &= \lambda \left(\bigcap_{j \in [s]} (\alpha_j^*)^{-1}(\mathcal{N}_{E_{i_j}}) \right) \\ &= \lambda(\{x \in \mathcal{E}_{\mathbb{N}_+} \mid \forall j \in [s], \alpha_j^*(x) \in \mathcal{N}_{E_{i_j}}\}), \end{aligned}$$

where $\alpha_j \in (\mathbb{N}_+)_{k_{i_j}}$ is such that $\text{im}(\alpha_j) = V_j$. Since the sets V_j have pairwise intersections of size at most ℓ , in the set above, the coordinates x_A with $|A| > \ell$ are only constrained by at most one of the α_j^* , so Fubini's Theorem gives

$$\mathbb{P}[\forall j \in [s], V_j \in E_{i_j}(\mathbf{K})] = \int_{\mathcal{E}_{V, \ell}} \prod_{j \in [s]} W_{\mathcal{N}_{E_{i_j}}}^{\ell}(\alpha_j^*(x)) \, d\lambda(x),$$

where $V \stackrel{\text{def}}{=} \bigcup_{j \in [s]} V_j$.

Since each ϕ^{I_i} satisfies $\text{CliqueDisc}[\ell]$, by Lemma 6.5, it follows that $W_{\mathcal{N}_{E_{i_i}}}^{\ell} = \phi^{I_i}(\rho_{k_{i_i}})$ a.e., so we get

$$\mathbb{P}[\forall j \in [s], V_j \in E_{i_j}(\mathbf{K})] = \prod_{j \in [s]} \phi^{I_{i_j}}(\rho_{k_{i_j}}) = \prod_{j \in [s]} \mathbb{P}[V_j \in E_{i_j}(\mathbf{K})],$$

as desired. $\square$

Proposition 6.7 (and Theorem 3.14) will be sufficient to handle the case in the definition of symmetric ℓ -locality when all sets have size at least ℓ . For smaller sets, we need the notion of categoricity of elements of $\text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ defined below.

Definition 6.8. For $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$, let $\text{Th}(\phi)$ be the theory obtained from T by adding the axiom $\forall \vec{x}, \neg D_{\text{open}}(M)(\vec{x})$ for every $M \in \mathcal{M}[T]$ such that $\phi(M) = 0$, i.e., it is the theory whose models are precisely the ones that have positive density in ϕ .

Recall that in model theory a theory T is called ℓ -categorical if it has exactly one model of size ℓ up to isomorphism. We say that $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ is ℓ -categorical if $\text{Th}(\phi)$ is ℓ -categorical.

Remark 2. Since $\sum_{M \in \mathcal{M}_\ell[T]} \phi(M) = 1$, it follows that ϕ is ℓ -categorical if and only if $\phi(M) \in \{0, 1\}$ for every $M \in \mathcal{M}_\ell[T]$.

Lemma 6.9. Let $I: T_1 \rightsquigarrow T_2$ be an open interpretation and let $\phi \in \text{Hom}^+(\mathcal{A}[T_2], \mathbb{R})$ be ℓ -categorical. Then ϕ^I is ℓ -categorical.

Proof. Since for $M \in \mathcal{M}_\ell[T_1]$, we have $\phi^I(M) = \sum \{\phi(N) \mid N \in \mathcal{M}_\ell[T_2] \wedge I(N) \cong M\}$, it follows that $\phi^I(M) > 0$ if and only if $M \cong I(N_0)$ for the unique model $N_0 \in \mathcal{M}_\ell[\text{Th}(\phi)]$. $\square$

Lemma 6.10. If $\phi \in \text{Hom}^+(\mathcal{A}[T_{k\text{-Hypergraph}}], \mathbb{R})$ is ℓ -categorical for $\ell \geq k$ then $\phi(\rho_k) \in \{0, 1\}$, that is, the hypergraphon ϕ is either empty or complete.

Proof. Let M be the unique k -hypergraph on ℓ vertices such that $\phi(M) = 1$. Then $M \in \{K_\ell^{(k)}, \overline{K}_\ell^{(k)}\}$ as $\phi(K_\ell^{(k)}) = \phi(\overline{K}_\ell^{(k)}) = 0$ would have contradicted Ramsey's Theorem. The lemma follows. $\square$

Lemma 6.11. If $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ is ℓ -categorical and $0 \leq \ell' \leq \ell$, then ϕ is ℓ' -categorical.

Proof. Let $M \in \mathcal{M}_{\ell'}[T]$ and consider the open interpretation $I: T_{\ell'\text{-Hypergraph}} \rightsquigarrow T$ that declares m -edges to be isomorphic copies of M . By Lemma 6.9, it follows that ϕ^I is ℓ -categorical, and it follows from Lemma 6.10 that ϕ^I is either the empty or the complete hypergraphon. Now, ϕ is ℓ' -categorical by Remark 2. $\square$

Lemma 6.12. If $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ satisfies $\text{UInduce}[\ell]$, then ϕ is ℓ' -categorical for every $0 \leq \ell' \leq \ell$.

Proof. By Lemma 6.11, it is enough to show the case $\ell' = \ell$. Let $I: T \rightsquigarrow T \cup T_{\ell\text{-Hypergraph}}$ and $J: T_{\ell\text{-Hypergraph}} \rightsquigarrow T \cup T_{\ell\text{-Hypergraph}}$ be the structure-erasing

interpretations. Let $\mathcal{N}$ be a T -on such that $\phi_{\mathcal{N}} = \phi$ and for $M \in \mathcal{M}_{\ell}[T]$, let $\mathcal{H}$ be the $T \cup T_{\ell\text{-Hypergraph}}$ -on given by

$$\mathcal{H}_P \stackrel{\text{def}}{=} \mathcal{N}_P; \quad \mathcal{H}_E \stackrel{\text{def}}{=} \bigcup_{\substack{K \in \mathcal{K}_{\ell}[T] \\ K \cong M}} T_{\text{ind}}(K, \mathcal{N})$$

for every predicate symbol P in the language of T .

Let $\widehat{M} \in \mathcal{M}_{\ell}[T \cup T_{\ell\text{-Hypergraph}}]$ be such that $I(\widehat{M}) \cong M$ and $J(\widehat{M}) \cong \rho_{\ell}$. Then

$$\phi(M) = \phi_{\mathcal{H}}(\widehat{M}) = \phi(M)\phi_{\mathcal{H}}^J(\rho_{\ell}) = \phi(M)^2,$$

where the second equality follows since $\phi \in \mathbf{UInduce}[\ell]$. Hence $\phi(M) \in \{0, 1\}$ for every $M \in \mathcal{M}_{\ell}[T]$, so ϕ is ℓ -categorical by Remark 2. $\square$

Remark 3. The converse to Lemma 6.12 is very far from being true. For example, every graphon is 1-categorical, and, slightly less trivially, every tournamon is 2-categorical. They are seldom uniquely 1-inducible.

We can finally prove the last implication of Theorem 3.11.

Lemma 6.13 (Theorem 3.11(i) $\implies$ (iii)). *If $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ satisfies $\mathbf{UInduce}[\ell]$, then ϕ is symmetrically ℓ -local.*

Proof. Let $\mathbf{K}$ be the exchangeable array corresponding to ϕ . We need to show that for every finite collection $(V_i)_{i \in [t]}$ of finite subsets of $\mathbb{N}_+$ with pairwise intersections of size at most ℓ and every collection $(M_i)_{i \in [t]}$ of models of T , we have

$$\mathbb{P}[\forall i \in [t], \mathbf{K}|_{V_i} \cong M_i] = \prod_{i \in [t]} \mathbb{P}[\mathbf{K}|_{V_i} \cong M_i].$$

By Lemma 6.12, we know that ϕ is ℓ' -categorical for every $0 \leq \ell' \leq \ell$, which implies that if $|V| \leq \ell$, then $\mathbb{P}[\mathbf{K}|_V \cong M] = \phi(M) \in \{0, 1\}$, i.e., the event $\mathbf{K}|_V \cong M$ is trivial. So we may assume that $|V_i| > \ell$ for every $i \in [t]$.

Let $\vec{k} = (k_1, \dots, k_t)$ be given by $k_i \stackrel{\text{def}}{=} |V_i|$ and consider the interpretation $I: T_{\vec{k}\text{-Hypergraph}} \rightsquigarrow T$ that declares E_i -edges to be isomorphic copies of M_i . In other words, I is given by

$$I(E_i)(x_1, \dots, x_{k_i}) \stackrel{\text{def}}{=} \bigvee_{\sigma \in S_{k_i}} D_{\text{open}}(M_i)(x_{\sigma(1)}, \dots, x_{\sigma(k_i)}).$$

By Theorem 3.3, we know that for every $i \in [t]$ we have $\phi^{I \circ I_i} \in \mathbf{UInduce}[\ell]$ and by Theorem 3.14, it follows that $\phi^{I \circ I_i} \in \mathbf{CliqueDisc}[\ell]$. Then we have

$$\begin{aligned} \mathbb{P}[\forall i \in [t], \mathbf{K}|_{V_i} \cong M_i] &= \mathbb{P}[\forall i \in [t], V_i \in E_i(I(\mathbf{K}))] \\ &= \prod_{i \in [t]} \mathbb{P}[V_i \in E_i(I(\mathbf{K}))] = \prod_{i \in [t]} \mathbb{P}[\mathbf{K}|_{V_i} \cong M_i], \end{aligned}$$

where the second equality follows from Proposition 6.7. $\square$

We finish this section with the (now trivial) proof of Theorem 3.1.

Proof of Theorem 3.1. The facts $\mathbf{Independence}[\ell] \implies \mathbf{Independence}[\ell - 1]$ and $\mathbf{UCouple}[\ell] \implies \mathbf{UCouple}[\ell - 1]$ follow easily from definitions. The fact that $\mathbf{UInduce}[\ell] \implies \mathbf{UInduce}[\ell - 1]$ follows since symmetric ℓ -locality trivially implies symmetric $(\ell - 1)$ -locality and from Lemmas 6.3 and 6.13. $\square$

7 Unique coupleability

In this section we prove Theorem 3.10. We start with the equivalence (i) $\equiv$ (ii) $\equiv$ (iii). While implications (i) $\implies$ (iii) and (iii) $\implies$ (ii) are fairly straightforward, the proof of the implication (ii) $\implies$ (i) is more involved and naturally splits into five rather independent parts:

1. Show that unique coupleability of ϕ with the quasirandom ℓ' -hypergraphon $\psi_{\ell', p}$ for some $p \in (0, 1)$ implies the same statement for *every* $p \in (0, 1)$.
2. Show that unique coupleability of ϕ with the quasirandom ℓ' -hypergraphon $\psi_{\ell', p}$ for all $p \in (0, 1)$ implies that ϕ is unique coupleable with the quasirandom c -colored ℓ' -hypergraphon $\psi_{\ell', q}$ for every $c \geq 2$ and every $q \in \Pi_c$.
3. Show that unique coupleability of ϕ with all quasirandom colored ℓ' -hypergraphons for $\ell' \in [\ell]$ implies that ϕ is uniquely coupleable with all independent couplings $\psi_{1, p_1} \otimes \cdots \otimes \psi_{\ell, p_\ell}$ of quasirandom colored ℓ' -hypergraphons for $\ell' \in [\ell]$.
4. Show that in an arbitrary theory T' , the set of elements that are uniquely coupleable with $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ is closed in $\text{Hom}^+(\mathcal{A}[T'], \mathbb{R})$ in the L_1 -topology.

5. Show that for any pure canonical theory $T_{\mathcal{L}}$, the set of all elements of the form $(\psi_{1,p} \otimes \cdots \otimes \psi_{\ell,p})^I$, where $I: T_{\mathcal{L}} \rightsquigarrow \bigcup_{k=1}^{\ell} T_{c,k}$ is an open interpretation, is dense in the set of $\psi \in \text{Hom}^+(\mathcal{A}[T_{\mathcal{L}}], \mathbb{R})$ of rank at most ℓ (again in the L_1 -topology) and apply Theorem 3.3.

Let us point out that items 1, 2 and 3 combined show a strengthened version of implication (ii) $\implies$ (iii) that allows for multiple colors and arbitrary densities. Furthermore, most likely items 4 and 5 in this program can be replaced with an ad hoc argument but we prefer this more structured approach.

We start with item 1.

Lemma 7.1. *Let $\ell \in \mathbb{N}_+$ and $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$. If there exists $p \in (0, 1)$ such that ϕ is uniquely coupleable with the quasirandom ℓ -hypergraphon $\psi_{\ell,p}$, then ϕ is uniquely coupleable with $\psi_{\ell,q}$ for every $q \in (0, 1)$.*

Proof. Let $\mathcal{C}_q$ be the set of all couplings of ϕ with $\psi_{\ell,q}$. Our objective is to show that $|\mathcal{C}_q| = 1$. Without loss of generality, let us suppose that $p < q$ (otherwise, we can use the complementation automorphism $C: T_{\ell\text{-Hypergraph}} \rightsquigarrow T_{\ell\text{-Hypergraph}}$ given by $C(E)(\vec{x}) \stackrel{\text{def}}{=} \bigwedge_{1 \leq i < j \leq \ell} x_i \neq x_j \wedge \neg E(\vec{x})$ and Theorem 3.3). Intuitively, we are going to “dilute” $\psi_{\ell,q}$ by a factor $t = p/q$ so that it will turn into $\psi_{\ell,p}$. The simplest way to make this intuition precise is by introducing yet another quasirandom hypergraphon $\psi_{\ell,t}$ on the same ground set and then taking its intersection with $\psi_{\ell,q}$.

Formally, we consider the commutative diagram

$$\begin{array}{ccccc}
 & & T & & \\
 & \swarrow I & & \searrow I & \\
 T_{\ell\text{-Hypergraph}} & & & & T_{\ell\text{-Hypergraph}} \\
 \downarrow J & & & & \downarrow J \\
 T \cup T_{\ell\text{-Hypergraph}} & & & & T \cup T_{\ell\text{-Hypergraph}} \\
 & \searrow \text{id}_T \cup I' & & \swarrow & \\
 & T \cup T_{\ell\text{-Hypergraph}} \cup T_{\ell\text{-Hypergraph}} & & & \\
 & \uparrow \hat{J} & & & \\
 & T_{\ell\text{-Hypergraph}} \cup T_{\ell\text{-Hypergraph}} & & & \\
 \uparrow I' & & & & \uparrow
 \end{array}
 \tag{24}$$

where $I, J, \hat{J}$ and the unlabeled arrows are the structure-erasing interpretations, with the unlabeled arrows keeping the second copy of $T_{\ell\text{-Hypergraph}}$, and

I' is given by

$$I'(E)(x_1, \dots, x_\ell) = E(x_1, \dots, x_\ell) \wedge E'(x_1, \dots, x_\ell).$$

Here E corresponds to the first copy of T_ℓ -Hypergraph and E' corresponds to the second one.

We now define the dilution map $F: \mathcal{C}_q \rightarrow \mathcal{C}_p$ by

$$F(\xi) \stackrel{\text{def}}{=} (\xi \otimes \psi_{\ell,t})^{\text{id}_T \cup I'},$$

where $t \stackrel{\text{def}}{=} p/q \in (0, 1)$. The fact that $F(\xi)$ is indeed an element of $\mathcal{C}_p$ follows from

$$\begin{aligned} ((\xi \otimes \psi_{\ell,t})^{\text{id}_T \cup I'})^I &= (\phi \otimes \psi_{\ell,t})^I = \phi; \\ ((\xi \otimes \psi_{\ell,t})^{\text{id}_T \cup I'})^J &= (\psi_{\ell,q} \otimes \psi_{\ell,t})^{I'} = \psi_{\ell,p}. \end{aligned}$$

For $M \in \mathcal{M}[T]$ and $U \subseteq \binom{V(M)}{\ell}$, let M_U be the model of $T \cup T_\ell$ -Hypergraph obtained from M by declaring the ℓ -hypergraph edge set to be U , that is, we have $I(M_U) = M$ and $E(J(M_U)) = U$. Then we have

$$F(\xi)(\langle M_U \rangle) = t^{|U|} \sum_{\substack{W \subseteq \binom{[m]}{\ell} \\ U \subseteq W}} (1-t)^{|W \setminus U|} \xi(\langle M_W \rangle).$$

By Möbius inversion, it follows that F is injective⁶, hence $|\mathcal{C}_q| \leq |\mathcal{C}_p| = 1$ as claimed. $\square$

We now proceed to item 2 of our program.

Lemma 7.2. *Let $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ and $\ell \in \mathbb{N}_+$ and suppose that for every $p \in (0, 1)$, ϕ is uniquely coupleable with the quasirandom ℓ -hypergraphon $\psi_{\ell,p}$. Then for every $c \geq 2$ and every $q \in \Pi_c$, ϕ is uniquely coupleable with the quasirandom c -colored ℓ -hypergraphon $\psi_{\ell,q}$.*

⁶The left-inverse is given by

$$F^{-1}(\zeta)(\langle M_U \rangle) = t^{-|U|} \sum_{\substack{W \subseteq \binom{[m]}{\ell} \\ U \subseteq W}} \left(1 - \frac{1}{t}\right)^{|W \setminus U|} \zeta(\langle M_W \rangle).$$

Proof. For $i \in [c]$, consider the following commutative diagram

$$\begin{array}{ccccc}
T_{\ell}\text{-Hypergraph} & \xrightarrow{J} & T \cup T_{\ell}\text{-Hypergraph} & \xleftarrow{I} & T \\
I'_i \downarrow & & \text{id}_T \cup I'_i \downarrow & \swarrow I_c & \\
T_{c,\ell} & \xrightarrow{J_c} & T \cup T_{c,\ell} & &
\end{array}$$

where I , I_c , J and J_c are structure-erasing and I'_i is given by

$$I'_i(E)(x_1, \dots, x_\ell) \stackrel{\text{def}}{=} E_i(x_1, \dots, x_\ell).$$

The set $\mathcal{K}_m[T_{c,\ell}]$ of labeled models of size m can be naturally identified with functions $f: \binom{[m]}{\ell} \rightarrow [c]$: given $m \in \mathbb{N}$ and $f: \binom{[m]}{\ell} \rightarrow [c]$, $C_f \in \mathcal{K}_m[T_{c,\ell}]$ is given by

$$V(C_f) \stackrel{\text{def}}{=} [m]; \quad R_{E_i}(C_f) \stackrel{\text{def}}{=} \{\alpha \in ([m])_\ell \mid f(\text{im}(\alpha)) = i\} \quad (i \in [c]).$$

Let $F \stackrel{\text{def}}{=} C^{-1}$. Given further $K \in \mathcal{K}_m[T]$ and $f: \binom{[m]}{\ell} \rightarrow [c]$, let K_f be the alignment of K and C_f , that is, K_f is the unique model in $\mathcal{K}_m[T \cup T_{c,\ell}]$ such that $I_c(K_f) = K$ and $J_c(K_f) = C_f$. Similarly, given $U \subseteq \binom{[m]}{\ell}$, let $K_U \in \mathcal{K}_m[T \cup T_{\ell}\text{-Hypergraph}]$ be the unique model such that $I(K_U) = K$ and $R_E(K_U) = \{\alpha \in ([m])_\ell \mid \text{im}(\alpha) \in U\}$.

Let $\psi \stackrel{\text{def}}{=} \psi_{\ell,q} \in \text{Hom}^+(\mathcal{A}[T_{c,\ell}], \mathbb{R})$ and let ξ be a coupling of ϕ and ψ . Our goal is to show that

$$\xi(\langle K_f \rangle) = \psi(\langle C_f \rangle) \phi(\langle K \rangle) \quad (25)$$

for every $m \in \mathbb{N}$, every $K \in \mathcal{K}_m[T]$ and every $f: \binom{[m]}{\ell} \rightarrow [c]$. Note that to improve readability, here and in the forthcoming calculations, K and K_f are identified with their isomorphism classes $[K]$, $[K_f]$ in $\mathcal{M}_m$.

If $m < \ell$, then (25) holds trivially and if $\phi(\langle K \rangle) = 0$, then both sides of (25) are 0, so suppose $m \geq \ell$ and $\phi(\langle K \rangle) > 0$. Note that $\xi^{\text{id}_T \cup I'_i \circ J} = \psi^{I'_i} = \psi_{\ell,q_i} \in \text{Hom}^+(\mathcal{A}[T_{\ell}\text{-Hypergraph}], \mathbb{R})$, hence $\xi^{\text{id}_T \cup I'_i}$ is a coupling of ϕ and ψ_{ℓ,q_i} , so we must have $\xi^{\text{id}_T \cup I'_i} = \phi \otimes \psi_{\ell,q_i}$. Note also that for $m \in \mathbb{N}$, $K \in \mathcal{K}_m[T]$ and $U \subseteq \binom{[m]}{\ell}$, we have

$$\pi^{\text{id}_T \cup I'_i}(\langle K_U \rangle) = \sum_{\substack{f: \binom{[m]}{\ell} \rightarrow [c] \\ f^{-1}(i) = U}} \langle K_f \rangle. \quad (26)$$

Pick now $\mathbf{f}: \binom{[m]}{\ell} \rightarrow [c]$ at random according to the distribution

$$\mathbb{P}[\mathbf{f} = f] \stackrel{\text{def}}{=} \frac{\xi(\langle K_f \rangle)}{\phi(\langle K \rangle)}.$$

The identity (26) allows us to compute, for $A \in \binom{[m]}{\ell}$ and $i \in [c]$, that

$$\begin{aligned} \mathbb{P}[\mathbf{f}(A) = i] &= \sum_{\substack{f: \binom{[m]}{\ell} \rightarrow [c] \\ f(A) = i}} \frac{\xi(\langle K_f \rangle)}{\phi(\langle K \rangle)} = \sum_{\substack{U \subseteq \binom{[m]}{\ell} \\ A \in U}} \frac{\xi^{\text{id}_T \cup I'_i}(\langle K_U \rangle)}{\phi(\langle K \rangle)} \\ &= \sum_{\substack{U \subseteq \binom{[m]}{\ell} \\ A \in U}} q_i^{|U|} (1 - q_i)^{\binom{m}{\ell} - |U|} = q_i, \end{aligned}$$

where the the second equality follows from (26) and the third equality follows since $\xi^{\text{id}_T \cup I'_i} = \phi \otimes \psi_{\ell, q_i}$. Since $\psi(\langle C_f \rangle) = \prod_{A \in \binom{[m]}{\ell}} q_{f(A)}$, to complete the proof of (25), it remains to show that the values $(\mathbf{f}(A) \mid A \in \binom{[m]}{\ell})$ of $\mathbf{f}$ are mutually independent.

For that purpose, it is in turn sufficient to prove that for every fixed $A_0 \in \binom{[m]}{\ell}$ and every fixed $i_0 \in [c]$, the event $\mathbf{f}(A_0) = i_0$ is independent from $\mathbf{f}|_W$, where $W \stackrel{\text{def}}{=} \binom{[m]}{\ell} \setminus \{A_0\}$.

To do so, we will generate the distribution of $\mathbf{f}$ in a very specific way. Let $\mathcal{N}$ be a T -on such that $\phi = \phi_{\mathcal{N}}$ and note that $\psi_{\ell, q_{i_0}} = \phi_{\mathcal{N}'} \in \text{Hom}^+(\mathcal{A}[T_{\ell\text{-Hypergraph}}], \mathbb{R})$ for the $(\ell - 1)$ -independent $T_{\ell\text{-Hypergraph}}$ -on $\mathcal{N}'$ given by

$$\mathcal{N}'_E \stackrel{\text{def}}{=} \{x \in \mathcal{E}_{\ell} \mid x_{[\ell]} < q_{i_0}\}. \quad (27)$$

Since $\xi^{\text{id}_T \cup I'_{i_0}} = \phi \otimes \psi_{\ell, q_{i_0}} = \phi_{\mathcal{N} \otimes \mathcal{N}'}$, by Proposition 4.3 applied to the interpretation $\text{id}_T \cup I'_{i_0}$, there exists a $(T \cup T_{c, \ell})$ -on $\mathcal{H}$ over $[0, 1]^4$ such that $\phi_{\mathcal{H}} = \xi$ and

$$\begin{aligned} \mathcal{H}_P &= \mathcal{N}_P \times \mathcal{E}_{k(P)}([0, 1]^3) \text{ a.e.} \quad (P \in \mathcal{L}); \\ \mathcal{H}_{E_{i_0}} &= \mathcal{E}_{\ell} \times \mathcal{N}'_E \times \mathcal{E}_{\ell}([0, 1]^2) \text{ a.e.,} \end{aligned} \quad (28)$$

where $\mathcal{L}$ is the language of T .

Let now $(\boldsymbol{\theta}^1, \boldsymbol{\theta}^2, \boldsymbol{\theta}^3, \boldsymbol{\theta}^4)$ be picked at random in $\mathcal{E}_{\mathbb{N}_+}([0, 1]^4)$ according to λ and let $\mathbf{K}$ be the exchangeable array corresponding to $\mathcal{H}$ with respect to $(\boldsymbol{\theta}^1, \boldsymbol{\theta}^2, \boldsymbol{\theta}^3, \boldsymbol{\theta}^4)$. Denote also $\mathbf{F} \stackrel{\text{def}}{=} F(J_c(\mathbf{K}|_{[m]}))$; $\mathbf{F} = (\mathbf{F}(A_0), \mathbf{F}|_W)$, and let E be the event $I_c(\mathbf{K}|_{[m]}) = K$. Then the function $\mathbf{f}$ is equidistributed with

the function $\mathbf{F}$ conditioned by the event E . It remains to note that by (28), the event $\mathbf{F}(A_0) = i_0$ depends only on the coordinate $\theta_{A_0}^2$ (warning: we do *not* claim that the whole random variable $\mathbf{F}(A_0)$ depends only on $\theta_{A_0}^2$). On the other hand, both E and $\mathbf{F}|_W$ do not depend on it; more precisely, E depends only on θ^1 and $\mathbf{F}|_W$ depends on those θ_B^j with $j \in [4]$, $|B| \leq \ell$ and $B \neq A_0$. $\square$

We now address item 3 of our program (cf. the second remark made after the statement of Theorem 3.10).

Lemma 7.3. *Let $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ and $\psi_i \in \text{Hom}^+(\mathcal{A}[T_i], \mathbb{R})$ for $i \in [t]$. Let also $\ell_1 \leq \dots \leq \ell_t$ and suppose that the following hold.*

- i. *For every $i \in \{1, \dots, t-1\}$, we have $\text{rk}(\psi_i) \leq \ell_i$.*
- ii. *For every $i \in \{2, \dots, t\}$, we have $\psi_i \in \text{Independence}[\ell_{i-1}]$.*
- iii. *For every $i \in \{1, \dots, t\}$, ϕ and ψ_i are uniquely coupleable.*

Then $\phi, \psi_1, \dots, \psi_t$ are uniquely coupleable.

Proof. The proof is by induction on t . For $t = 1$, the result is trivial. For $t = 2$, let $I_i: T \cup T_i \rightsquigarrow T \cup T_1 \cup T_2$, $J_i: T_i \rightsquigarrow T \cup T_1 \cup T_2$ and $J: T \rightsquigarrow T \cup T_1 \cup T_2$ be the structure-erasing interpretations. Let $\mathcal{L}$, $\mathcal{L}_1$ and $\mathcal{L}_2$ be the languages of T , T_1 and T_2 , respectively. Let also $\mathcal{N}$ be a T -on with $\phi_{\mathcal{N}} = \phi$ and $\mathcal{H}^2$ be an ℓ_1 -independent T_2 -on with $\phi_{\mathcal{H}^2} = \psi_2$. Fix a coupling ξ of ϕ, ψ_1, ψ_2 . Since ϕ and ψ_2 are uniquely coupleable, we know that $\xi^{I_2} = \phi \otimes \psi_2 = \phi_{\mathcal{N} \otimes \mathcal{H}^2}$. By Proposition 4.3, there exists a $(T \cup T_1 \cup T_2)$ -on $\mathcal{G}$ over $[0, 1]^4$ such that $\phi_{\mathcal{G}} = \xi$ and

$$\mathcal{G}_P = \begin{cases} \mathcal{N}_P \times \mathcal{E}_{k(P)}([0, 1]^3), & \text{if } P \in \mathcal{L}; \\ \mathcal{E}_{k(P)} \times \mathcal{H}_P^2 \times \mathcal{E}_{k(P)}([0, 1]^2), & \text{if } P \in \mathcal{L}_2. \end{cases}$$

On the other hand, for the predicate symbols P in $\mathcal{L}_1$, by possibly changing zero-measure sets of the corresponding P -ons $\mathcal{G}_P$ using Proposition 4.2, we may suppose that $\text{rk}(J_1(\mathcal{G})) \leq \text{rk}(\psi_1) \leq \ell_1$.

Let us pick $\theta \stackrel{\text{def}}{=} (\theta^1, \theta^2, \theta^3, \theta^4)$ at random in $\mathcal{E}_{\mathbb{N}_+}([0, 1]^4)$ according to λ and let $\mathbf{K}$ be the exchangeable array corresponding to $\mathcal{G}$ with respect to θ . Then we know that $J(\mathbf{K})$ depends only on θ^1 , $J_1(\mathbf{K})$ depends only on $((\theta_A^1, \theta_A^2, \theta_A^3, \theta_A^4) \mid |A| \leq \ell_1)$ and $J_2(\mathbf{K})$ depends only on $(\theta_A^2 \mid |A| > \ell_1)$ (as

$\mathcal{H}^2$ is ℓ_1 -independent), so $J_2(\mathbf{K})$ is independent from $(J(\mathbf{K}), J_1(\mathbf{K}))$. This means that for every $m \in \mathbb{N}$ and every $K \in \mathcal{K}_m[T \cup T_1 \cup T_2]$, we have

$$\begin{aligned}
\xi(\langle K \rangle) &= \mathbb{P}[\mathbf{K}|_{[m]} = K] \\
&= \mathbb{P}[J(\mathbf{K})|_{[m]} = J(K) \wedge J_1(\mathbf{K})|_{[m]} = J_1(K) \wedge J_2(\mathbf{K})|_{[m]} = J_2(K)] \\
&= \mathbb{P}[J(\mathbf{K})|_{[m]} = J(K) \wedge J_1(\mathbf{K})|_{[m]} = J_1(K)] \cdot \mathbb{P}[J_2(\mathbf{K})|_{[m]} = J_2(K)] \\
&= \mathbb{P}[I_1(\mathbf{K})|_{[m]} = I_1(K)] \cdot \mathbb{P}[J_2(\mathbf{K})|_{[m]} = J_2(K)] \\
&= \xi^{I_1}(\langle I_1(K) \rangle) \cdot \psi_2(\langle J_2(K) \rangle) \\
&= \phi(\langle J(K) \rangle) \cdot \psi_1(\langle J_1(K) \rangle) \cdot \psi_2(\langle J_2(K) \rangle),
\end{aligned}$$

where the last equality follows since ϕ is uniquely coupleable with ψ_1 and ξ^{I_1} is a coupling of ϕ and ψ_1 . Therefore $\xi = \phi \otimes \psi_1 \otimes \psi_2$.

For the case $t \geq 3$, let $I: T \cup \bigcup_{i=2}^t T_i \rightsquigarrow T \cup \bigcup_{i=1}^t T_i$ be the structure-erasing interpretation and note that for a coupling ξ of $\phi, \psi_1, \dots, \psi_t$, it follows that ξ^I is a coupling of $\phi, \psi_2, \dots, \psi_t$. By inductive hypothesis, we must have $\xi^I = \phi \otimes \widehat{\psi}$, where $\widehat{\psi} \stackrel{\text{def}}{=} \bigotimes_{i=2}^t \psi_i$. In fact, since $\phi, \psi_2, \dots, \psi_t$ are uniquely coupleable, it also follows that ϕ is uniquely coupleable with $\widehat{\psi}$ (as any coupling of ϕ with $\widehat{\psi}$ can be seen as a coupling of $\phi, \psi_2, \dots, \psi_t$). But by Theorem 3.4, we know that $\widehat{\psi} \in \text{Independence}[\ell_1]$ and since ξ can also be seen as a coupling of $\phi, \psi_1, \widehat{\psi}$, we get $\xi = \phi \otimes \bigotimes_{i=1}^t \psi_i$ from the previous case. $\square$

Lemma 7.4. *Let $c \geq 2$, $p \in \Pi_c$ and $k \in \mathbb{N}_+$. Then the quasirandom c -colored k -hypergraphon $\psi_{k,p}$ satisfies $\text{Independence}[k-1]$ and $\text{rk}(\psi_{k,p}) = k$.*

Proof. Note that $\psi_{k,p}$ can be represented by the $T_{c,k}$ -on $\mathcal{N}^{k,p}$ given by

$$\mathcal{N}_{E_i}^{k,p} \stackrel{\text{def}}{=} \left\{ x \in \mathcal{E}_k \left| \sum_{j=1}^{i-1} p_j \leq x_{[k]} < \sum_{j=1}^i p_j \right. \right\} \quad (i \in [c]),$$

hence $\psi_{k,p} \in \text{Independence}[k-1]$ and $\text{rk}(\psi_{k,p}) \leq k$. Since $c \geq 2$, it follows that $\text{rk}(\psi_{k,p}) > 0$, so by Theorem 3.2 and Proposition 4.1, we must have $\text{rk}(\psi_{k,p}) = k$. $\square$

Proceeding to item 4 in the program, we introduce the L_1 -topology on theons that is a direct analogue of the L_1 -topology on graphons [Lov12, Sct. 8.2.5 and Sct. 8.3].

Definition 7.5. If T is a theory in a language $\mathcal{L}$ and $\phi_1, \phi_2 \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$, then the L_1 -distance between ϕ_1 and ϕ_2 is defined as

$$\delta_1(\phi_1, \phi_2) \stackrel{\text{def}}{=} \min_{\mathcal{N}^1, \mathcal{N}^2} \sum_{P \in \mathcal{L}} \mu(\mathcal{N}_P^1 \triangle \mathcal{N}_P^2), \quad (29)$$

where the minimum is taken over T -ons $\mathcal{N}^1$ and $\mathcal{N}^2$ over the same space such that $\phi_1 = \phi_{\mathcal{N}^1}$ and $\phi_2 = \phi_{\mathcal{N}^2}$.

It is not immediately clear from this definition that the minimum in (29) is actually attained, nor is it clear why δ_1 is a metric.

The first issue is easy to address by giving an alternative purely algebraic definition. Namely, for any $P \in \mathcal{L}$ introduce the element $d_P \in \mathcal{A}[T \cup T]$ as

$$d_P \stackrel{\text{def}}{=} \sum_{\substack{K \in \mathcal{K}_{k(P)}[T \cup T] \\ \text{id}_{k(P)} \in R_{P_1}(K) \triangle R_{P_2}(K)}} \langle K \rangle,$$

where P_1 and P_2 are the two copies of P in $\mathcal{L} \dot{\cup} \mathcal{L}$, and let

$$d_T \stackrel{\text{def}}{=} \sum_{P \in \mathcal{L}} d_P.$$

This element measures the distance in a coupling of ϕ_1, ϕ_2 so we have

$$\delta_1(\phi_1, \phi_2) = \inf_{\xi} \xi(d_T), \quad (30)$$

where ξ runs over all couplings of ϕ_1 and ϕ_2 . Their set is determined in $\text{Hom}^+(\mathcal{A}[T \cup T], \mathbb{R})$ by countably many linear equations and hence compact. Therefore the minimum in (30) and (29) is actually achieved.

The second issue is trickier, and the proof is similar to the analogous proof that δ_1 is a metric in the case of graphons. Fortunately, we already did most of the necessary (and notationally heavy) work in the proof of Proposition 4.3; we defer the remaining part to Appendix A.

Let us finally remark why we need L_1 -topology at all instead of the standard and much nicer density topology (i.e., the one induced by the inclusion $\text{Hom}^+(\mathcal{A}[T], \mathbb{R}) \subseteq [0, 1]^{\mathcal{M}[T]}$ from the product topology). One simple explanation is that the set of all $\psi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ that are uniquely coupleable with some $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ is not closed in the latter.

Example 1. Let $\phi_p \in \text{Hom}^+(\mathcal{A}[T_{\text{Graph}}], \mathbb{R})$ be the quasirandom graphon of density $p \in (0, 1)$. If $(G_n)_{n \in \mathbb{N}}$ ($G_n \in \mathcal{M}_n[T_{\text{Graph}}]$) is a sequence of graphs converging to ϕ_p , then the associated step functions ψ_n converge to ϕ_p in the density topology. Since $\text{rk}(\psi_n) = 1$ and $\phi_p \in \text{Independence}[1]$, it follows that ϕ_p and ψ_n are uniquely coupleable, but $\phi_p = \lim_{n \rightarrow \infty} \psi_n$ is obviously not uniquely coupleable with itself.

The example above illustrates another crucial difference between the L_1 -topology and density topology: rank is lower semi-continuous in the former but not the latter. In fact, for pure canonical theories $T_{\mathcal{L}}$, the set $\{\psi \in \text{Hom}^+(\mathcal{A}[T_{\mathcal{L}}], \mathbb{R}) \mid \text{rk}(\psi) \leq r\}$ is closed in L_1 -topology but dense in $\text{Hom}^+(\mathcal{A}[T_{\mathcal{L}}], \mathbb{R})$ in density topology (if $r \geq 1$).

Lemma 7.6. *Let $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ and T' be an arbitrary theory. Then the set of $\psi \in \text{Hom}^+(\mathcal{A}[T'], \mathbb{R})$ that are uniquely coupleable with ϕ is closed in the L_1 -topology.*

Proof. Let $(\psi_n)_{n \in \mathbb{N}}$ be a sequence in $\text{Hom}^+(\mathcal{A}[T'], \mathbb{R})$ converging to ψ in the L_1 -topology and suppose every ψ_n is uniquely coupleable with ϕ . It is clear from the definition that $\delta_1(\phi \otimes \psi_n, \phi \otimes \psi) = \delta_1(\psi_n \otimes \psi)$, so $\phi \otimes \psi_n$ also converges to $\phi \otimes \psi$ in the L_1 -topology. For each $n \in \mathbb{N}$, let ζ_n be a coupling of ψ and ψ_n attaining the L_1 -distance in (30).

Let ξ be a coupling of ϕ and ψ ; we have to show that $\xi = \phi \otimes \psi$. Let $I: T' \cup T' \rightsquigarrow T \cup T' \cup T'$ and $J_i: T' \rightsquigarrow T' \cup T'$ be the structure-erasing interpretations, where J_i keeps the i -th copy of T' . Since ξ is a coupling of ϕ and $\psi = \zeta_n^{J_1}$, by Proposition 5.2, there exists a coupling ξ_n of ϕ and ζ_n such that $\widehat{\xi_n^{\text{id}_T \cup J_1}} = \xi$. Note that $\widehat{\xi_n}$ can also be seen as a coupling of ϕ , ψ and ψ_n as $\widehat{\xi_n^I} = \zeta_n$.

Let now $\mathcal{N}^n$ be a $(T \cup T' \cup T')$ -on such that $\widehat{\xi_n} = \phi_{\mathcal{N}^n}$. By considering the $(T \cup T')$ -ons $(\text{id}_T \cup J_1)(\mathcal{N}^n)$ and $(\text{id}_T \cup J_2)(\mathcal{N}^n)$, since ψ_n is uniquely coupleable with ϕ , we conclude from (29) that

$$\delta_1(\xi, \phi \otimes \psi_n) \leq \sum_{P \in \mathcal{L}'} \lambda(J_1(I(\mathcal{N}^n))_P \triangle J_2(I(\mathcal{N}^n))_P) = \zeta_n(d_{T'}) = \delta_1(\psi, \psi_n),$$

where $\mathcal{L}'$ is the language of T' . Since $\psi_n \rightarrow \psi$ and $\phi \otimes \psi_n \rightarrow \phi \otimes \psi$ in the L_1 -topology, it follows that $\xi = \phi \otimes \psi$. $\square$

We proceed to the last item 5 in our program, which is to provide a way of approximating Euclidean structures with interpretations of independent

couplings $\psi_{1,p} \otimes \cdots \otimes \psi_{\ell,p}$ of quasirandom colored hypergraphons in the L_1 -topology.

Lemma 7.7. *Let $\mathcal{L}$ be a language, $\phi \in \text{Hom}^+(\mathcal{A}[T_{\mathcal{L}}], \mathbb{R})$, $r \stackrel{\text{def}}{=} \text{rk}(\phi)$ and $\epsilon > 0$. Then there exist $c \geq 2$, $p \in \Pi_c$ and an open interpretation $I: T_{\mathcal{L}} \rightsquigarrow \bigcup_{k=1}^r T_{c,k}$ such that $\delta_1(\phi, (\bigotimes_{k=1}^r \psi_{k,p})^I) \leq \epsilon$.*

Proof. Let $\mathcal{N}$ be a $T_{\mathcal{L}}$ -on such that $\phi_{\mathcal{N}} = \phi$ and $\text{rk}(\mathcal{N}) = r$, that is, for each $P \in \mathcal{L}$, there exists $\mathcal{H}_P \subseteq \mathcal{E}_{k(P),r}$ such that $\mathcal{N}_P = \mathcal{H}_P \times [0, 1]^{\binom{[k(P)]}{>r}}$. By standard measure theory arguments, for each $P \in \mathcal{L}$, there exists a finite family of pairwise disjoint closed cubes $(C_j^P)_{j=1}^{m_P}$ ($C_j^P \subseteq \mathcal{E}_{k(P),r}$) such that setting $\mathcal{H}'_P \stackrel{\text{def}}{=} \bigcup_{j=1}^{m_P} C_j^P$ gives $\lambda(\mathcal{H}_P \triangle \mathcal{H}'_P) \leq \epsilon/|\mathcal{L}|$.

Let X be the set of all coordinates of vertices of all cubes C_j^P for all $P \in \mathcal{L}$. The set X induces a partition of $[0, 1]$ into intervals $J_1, \dots, J_c$ of positive length (we can ensure $c \geq 2$ by including an extra point if necessary). Define then $p \in \Pi_c$ by letting $p_i \stackrel{\text{def}}{=} \lambda(J_i) > 0$ and define the $(\bigcup_{k=1}^r T_{c,k})$ -on $\mathcal{G}$ by

$$\mathcal{G}_{E_i^k} \stackrel{\text{def}}{=} \{x \in \mathcal{E}_k \mid x_{[k]} \in J_i\} \quad (i \in [c], k \in [r]),$$

where for each $k \in [r]$, the symbols $E_1^k, \dots, E_c^k$ correspond to $T_{c,k}$.

Let $\psi \stackrel{\text{def}}{=} \phi_{\mathcal{G}}$ and note that ψ is a coupling of $\psi_{1,p}, \dots, \psi_{r,p}$, so we must have $\psi = \bigotimes_{k=1}^r \psi_{k,p}$ by Lemmas 7.3 and 7.4.

Note now that from the definition of X , each cube $C_j^P \subseteq \mathcal{E}_{k(P),r}$ can be written as a finite union of the form $\bigcup_{u \in U_{P,j}} \prod_{A \in r(k(P),r)} J_{i_{P,u,A}}$. We then define $I: T_{\mathcal{L}} \rightsquigarrow \bigcup_{k=1}^r T_{c,k}$ by

$$I(P)(x_1, \dots, x_{k(P)}) \stackrel{\text{def}}{=} \bigvee_{j=1}^{m_P} \bigvee_{u \in U_{P,j}} \bigwedge_{A \in r(k(P),r)} E_{i_{P,u,A}}^k(x_{\iota_A(1)}, \dots, x_{\iota_A(|A|)}) \quad (P \in \mathcal{L}).$$

Our definition ensures that

$$\begin{aligned} I(\mathcal{G})_P &= \bigcup_{j=1}^{m_P} \bigcup_{u \in U_{P,j}} \left(\prod_{A \in r(k(P),r)} J_{i_{P,u,A}} \times [0, 1]^{\binom{[k(P)]}{>r}} \right) \\ &= \bigcup_{j=1}^{m_P} (C_j^P \times [0, 1]^{\binom{[k(P)]}{>r}}) = \mathcal{H}'_P \times [0, 1]^{\binom{[k(P)]}{>r}}. \end{aligned}$$

This implies that

$$\delta_1(\phi, \psi) \leq \sum_{P \in \mathcal{L}} \lambda(\mathcal{N}_P \triangle (\mathcal{H}'_P \times [0, 1]^{(\lfloor \frac{k(P)}{r} \rfloor)})) = \sum_{P \in \mathcal{L}} \lambda(\mathcal{H}_P \triangle \mathcal{H}'_P) \leq \epsilon,$$

as desired. $\square$

We now have all the ingredients to show the equivalence (i) $\equiv$ (ii) $\equiv$ (iii) of Theorem 3.10.

Lemma 7.8 (Theorem 3.10(i) $\equiv$ (ii) $\equiv$ (iii)). *Let $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ and $\ell \in \mathbb{N}_+$. Then the following are equivalent.*

- i. $\phi \in \text{UCouple}[\ell]$.
- ii. For every $\ell' \in [\ell]$, there exists $p \in (0, 1)$ such that ϕ is uniquely coupleable with the quasirandom ℓ' -hypergraphon $\psi_{\ell', p}$.
- iii. There exist $p_1, \dots, p_\ell \in (0, 1)$ such that ϕ is uniquely coupleable with the independent coupling $\psi_{1, p_1} \otimes \dots \otimes \psi_{\ell, p_\ell}$ of quasirandom ℓ' -hypergraphons $\psi_{\ell', p_{\ell'}}$ for $\ell' \in [\ell]$.

Proof. Since ℓ' -hypergraphons have rank at most ℓ' , by Proposition 4.2, we have $\text{rk}(\psi_{1, p_1} \otimes \dots \otimes \psi_{\ell, p_\ell}) \leq \ell$, so the implication (i) $\implies$ (iii) follows.

Implication (iii) $\implies$ (ii) follows from Theorem 3.3 by considering the structure-erasing interpretations $I_k: T_k\text{-Hypergraph} \rightsquigarrow \bigcup_{\ell'=1}^{\ell} T_{\ell'}\text{-Hypergraph}$.

For the non-trivial implication (ii) $\implies$ (i), we want to show that ϕ is uniquely coupleable with any $\psi \in \text{Hom}^+(\mathcal{A}[T'], \mathbb{R})$ of rank at most ℓ . We can assume w.l.o.g. that $T' = T_{\mathcal{L}}$ for some language $\mathcal{L}$. Using Lemma 7.7, for each $n \in \mathbb{N}$, we can find $c_n \geq 2$, $p_n \in \Pi_{c_n}$ and $I_n: T_{\mathcal{L}} \rightsquigarrow \bigcup_{k=1}^{c_n} T_{c_n, k}$ such that $\delta_1(\phi, (\bigotimes_{k=1}^{c_n} \psi_{k, p_n})^{I_n}) \leq 1/n$.

By Lemmas 7.1, 7.2, 7.3 and 7.4, we know that ϕ is uniquely coupleable with $\bigotimes_{k=1}^{c_n} \psi_{k, p_n}$ and by Theorem 3.3, it follows that ϕ is also uniquely coupleable with $(\bigotimes_{k=1}^{c_n} \psi_{k, p_n})^{I_n}$.

Finally, since $((\bigotimes_{k=1}^{c_n} \psi_{k, p_n})^{I_n})_{n \in \mathbb{N}}$ converges to ψ in the L_1 -topology, by Lemma 7.6, it follows that ϕ is uniquely coupleable with ψ . $\square$

We now proceed to add items (vi) and (vii) to the list of equivalent properties of Theorem 3.10 (recall that (i) $\equiv$ (iv) $\equiv$ (v) and (iv) $\implies$ (vi) were proved in Section 4).

Lemma 7.9 (Theorem 3.10(vi) $\implies$ (vii)). *If $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ is ℓ -local, then $\phi \otimes \psi_{\text{lin}}$ satisfies $\text{UInduce}[\ell]$.*

Proof. By Lemma 6.3, it is enough to show that $\phi \otimes \psi_{\text{lin}}$ is symmetrically ℓ -local. Let $\mathbf{K}$ be the exchangeable array corresponding to $\phi \otimes \psi_{\text{lin}}$, and fix a finite family of finite sets $(V_i)_{i \in [t]}$ ($V_i \subseteq \mathbb{N}_+$) with pairwise intersections of size at most ℓ . We let $\mathbf{K}_i \stackrel{\text{def}}{=} \mathbf{K}|_{V_i} \in \mathcal{K}_{V_i}[T \cup T_{\text{LinOrder}}]$ and let $\mathbf{M}_i \stackrel{\text{def}}{=} [\mathbf{K}_i] \in \mathcal{M}_{|V_i|}[T \cup T_{\text{LinOrder}}]$ be the isomorphism type of $\mathbf{K}_i$. We have to prove that $\mathbf{M}_1, \dots, \mathbf{M}_t$ are mutually independent, and for that purpose we are going to apply Claim 4.8 again.

More specifically, let $I: T \rightsquigarrow T \cup T_{\text{LinOrder}}$ be the structure-erasing interpretation and $\mathbf{L}_i = I(\mathbf{K}_i) \in \mathcal{K}_{V_i}[T]$ be the results of erasing linear order. Likewise, let $J: T_{\text{LinOrder}} \rightsquigarrow T \cup T_{\text{LinOrder}}$, and let $\leq_i = J(\mathbf{K}_i)$ be the corresponding (random) linear order on V_i so that $\mathbf{K}_i = (\mathbf{L}_i, \leq_i)$. In Claim 4.8, we set $\mathbf{X} = (\leq_1, \dots, \leq_n)$, $\mathbf{Y}_i = \mathbf{L}_i$, and let $f_i(\leq_1, \dots, \leq_n, L_i)$ be the function first computing K_i from L_i and $\leq_i$ and then taking its isomorphism type $M_i = [K_i]$.

We know that the tuple $(\mathbf{L}_1, \dots, \mathbf{L}_t)$ is independent from $\mathbf{X} = (\leq_1, \dots, \leq_t)$ (as the coupling of ϕ and ψ_{lin} is independent) and that $\mathbf{L}_1, \dots, \mathbf{L}_t$ are mutually independent (as ϕ is ℓ -local). This gives us the first assumption in Claim 4.8: $\mathbf{X}, \mathbf{Y}_1, \dots, \mathbf{Y}_n$ are mutually independent (note that we do *not* claim that $\leq_1, \dots, \leq_n$ are mutually independent, this is in general not true). It remains to show that $(\mathbf{M}_1, \dots, \mathbf{M}_n)$ is independent from $(\leq_1, \dots, \leq_n)$, and it essentially follows from the observation that the function $f_i(X, Y_i)$ becomes invertible after fixing its first argument.

More specifically, we compute $\mathbf{L}_i = g_i(\leq_i, \mathbf{M}_i)$, where $g_i(\leq_i, M_i)$ is obtained by first aligning the internal order of $V(M_i)$ with the order $\leq_i$ on V_i , and then discarding it. The crucial property is that $L_i = g_i(\leq_i, M_i)$ if and only if $M_i = f_i((\leq_1, \dots, \leq_n), L_i)$. Using this, fixing arbitrary models $M_i \in \mathcal{M}_{|V_i|}[T \cup T_{\text{LinOrder}}]$ and a particular tuple of values $(\leq_1, \dots, \leq_t)$, we have the calculation

$$\begin{aligned} & \mathbb{P}[\forall i \in [t], \mathbf{M}_i \cong M_i \mid \forall i \in [t], \leq_i = \leq_i] \\ &= \mathbb{P}[\forall i \in [t], \mathbf{L}_i = g_i(\leq_i, M_i) \mid \forall i \in [t], \leq_i = \leq_i] \\ &= \mathbb{P}[\forall i \in [t], \mathbf{L}_i = g_i(\leq_i, M_i)] \\ &= \mathbb{P}[\forall i \in [t], \mathbf{M}_i \cong M_i]. \end{aligned}$$

This shows that $(\mathbf{M}_1, \dots, \mathbf{M}_t)$ is indeed independent from $(\leq_1, \dots, \leq_t)$. We are now in position to apply Claim 4.8 which completes the proof. $\square$

Lemma 7.10 (Theorem 3.10(vii) $\implies$ (ii)). *If the independent coupling of $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ with ψ_{lin} satisfies $\text{UInduce}[\ell]$, then for every $\ell' \in [\ell]$, ϕ is uniquely coupleable with the quasirandom ℓ' -hypergraphon $\psi_{\ell', 1/2} \in \text{Hom}^+(\mathcal{A}[T_{\ell'}\text{-Hypergraph}], \mathbb{R})$.*

Proof. Let $\mathcal{L}$ be the language of T and note that since $\text{UInduce}[\ell]$ implies $\text{UInduce}[\ell']$ (Theorem 3.1), it is sufficient to consider the case $\ell' = \ell$. Let us first assume $\ell \geq 2$.

Note that ψ_{lin} can be represented by the T_{LinOrder} -on $\mathcal{N}^<$ given by

$$\mathcal{N}^< \stackrel{\text{def}}{=} \{x \in \mathcal{E}_2 \mid x_{\{1\}} < x_{\{2\}}\},$$

and that $\psi_{\ell, 1/2}$ can be represented as

$$\mathcal{N}_E \stackrel{\text{def}}{=} \{x \in \mathcal{E}_\ell \mid x_{[\ell]} \leq 1/2\}.$$

Let ξ be a coupling of ϕ and $\psi_{\ell, 1/2}$ and let $\mathcal{N}$ be a $(T \cup T_{\ell}\text{-Hypergraph})$ -on such that $\phi_{\mathcal{N}} = \xi$. As in the proof of Lemma 7.2, for every $m \in \mathbb{N}$ and every $U \subseteq \binom{[m]}{\ell}$, let $H_U \in \mathcal{K}_m[T_{\ell}\text{-Hypergraph}]$ be the hypergraph given by $V(H_U) \stackrel{\text{def}}{=} [m]$ and $R_E(H_U) \stackrel{\text{def}}{=} \{\alpha \in ([m])_\ell \mid \text{im}(\alpha) \in U\}$. If we are further given $K \in \mathcal{K}_m[T]$, let $K_U \in \mathcal{K}_m[T \cup T_{\ell}\text{-Hypergraph}]$ be the alignment of K and H_U , that is, we have $R_P(K_U) \stackrel{\text{def}}{=} R_P(K)$ ($P \in \mathcal{L}$) and $R_E(K_U) \stackrel{\text{def}}{=} R_E(H_U)$. Finally, we let $K_U^< \in \mathcal{K}_m[T \cup T_{\ell}\text{-Hypergraph} \cup T_{\text{LinOrder}}]$ be the model obtained from K_U by equipping it with the natural order of $[m]$. Note that while we do need labels in K to properly define the models K_U and $K_U^<$, in the computations below they are treated as unlabeled models $[K_U]$, $[K_U^<]$, i.e., labels are discarded.

To show that ξ is the independent coupling of ϕ and $\psi_{\ell, 1/2}$, we need to show that for every $m \in \mathbb{N}$, every $K \in \mathcal{K}_m[T]$ and every $U \subseteq \binom{[m]}{\ell}$, we have

$$\xi(\langle K_U \rangle) = \phi(\langle K \rangle) \cdot \psi_{\ell, 1/2}(\langle H_U \rangle) = \frac{\phi(\langle K \rangle)}{2^{\binom{m}{\ell}}}. \quad (31)$$

The assertion is trivial if $m < \ell$, so suppose $m \geq \ell$. Fix $U \subseteq \binom{[m]}{\ell}$ and for every $v \in [m]$, define

$$V_v \stackrel{\text{def}}{=} \begin{cases} \left[\frac{v-1}{m}, \frac{v}{m} \right), & \text{if } v < m; \\ \left[\frac{m-1}{m}, 1 \right], & \text{if } v = m. \end{cases}$$

For $n \in \mathbb{N}$ and $y \in \mathcal{E}_n$, let $\alpha_y: [n] \rightarrow [m]$ be the unique function such that $y_{\{j\}} \in V_{\alpha_y(j)}$ for every $j \in [n]$. Finally, define the set

$$W_U \stackrel{\text{def}}{=} \left\{ (x, y) \in \mathcal{E}_\ell \times \mathcal{E}_\ell \mid |\text{im}(\alpha_y)| = \ell \wedge \left(x_{[\ell]} \leq \frac{1}{2} \equiv \text{im}(\alpha_y) \in U \right) \right\};$$

clearly, W_U is S_ℓ -invariant. This means that we can define the $(T \cup T_{\ell\text{-Hypergraph}} \cup T_{\text{LinOrder}})$ -on $\mathcal{H}^U$ over $[0, 1]^2$ by

$$\mathcal{H}_P^U \stackrel{\text{def}}{=} \mathcal{N}_P \times \mathcal{E}_{k(P)} \quad (P \in \mathcal{L}), \quad \mathcal{H}_\prec^U \stackrel{\text{def}}{=} \mathcal{E}_2 \times \mathcal{N}^<, \quad \mathcal{H}_E^U \stackrel{\text{def}}{=} W_U.$$

Obviously, if $(x, y) \in T_{\text{ind}}(K_m^{(\ell)}, W_U)$, then each $y_{\{j\}}$ must belong to a different V_v . Indeed, if there exist $j_1, j_2 \in [m]$ with $y_{\{j_1\}}, y_{\{j_2\}} \in V_v$ but $j_1 \neq j_2$, since $m \geq \ell \geq 2$, there exists $\beta \in ([m]_\ell)$ with $j_1, j_2 \in \text{im}(\beta)$ and thus $(x, y) \notin (\beta^*)^{-1}(W_U)$, a contradiction.

Our claim and the definition of W_U then imply

$$T_{\text{ind}}(K_m^{(\ell)}, W_U) = \left\{ (x, y) \in \mathcal{E}_m \times \mathcal{E}_m \mid |\text{im}(\alpha_y)| = m \wedge \bigwedge_{\beta \in ([m]_\ell)} (\beta^*(x) \in \mathcal{N}_E \equiv \text{im}(\alpha_{\beta^*(y)}) \in U) \right\}.$$

Thus, denoting by $J_\ell: T_{\ell\text{-Hypergraph}} \rightsquigarrow T \cup T_{\ell\text{-Hypergraph}} \cup T_{\text{LinOrder}}$ the structure-erasing interpretation, we get

$$\phi_{J_\ell(\mathcal{H}^U)}(K_m^{(\ell)}) = \frac{m!}{m^m} \psi_{\ell, 1/2}(H_U) = \frac{m!}{m^m \cdot 2^{\binom{m}{\ell}}}. \quad (32)$$

Let now $J: T \rightsquigarrow T \cup T_{\ell\text{-Hypergraph}} \cup T_{\text{LinOrder}}$ be another structure-erasing interpretation; we have

$$\begin{aligned} T_{\text{ind}}(K_{\binom{[m]}{\ell}}^<, \mathcal{H}^U) &= T_{\text{ind}}(K, J(\mathcal{H}^U)) \cap T_{\text{ind}}(K_m^{(\ell)}, J_\ell(\mathcal{H}^U)) \cap \{(x, y) \in \mathcal{E}_m \times \mathcal{E}_m \mid y_{\{1\}} < \dots < y_{\{m\}}\} \\ &= \{(x, y) \in \mathcal{E}_m \times \mathcal{E}_m \mid x \in T_{\text{ind}}(K_U, \mathcal{N}) \wedge \forall v \in [m], y_{\{v\}} \in V_v\}. \end{aligned}$$

Since $\phi_{\mathcal{N}} = \xi$, we get

$$\xi(\langle K_U \rangle) = m^m \cdot \phi_{\mathcal{H}^U}(\langle K_{\binom{[m]}{\ell}}^{\leq} \rangle) = \frac{m^m \cdot \phi(\langle K \rangle) \cdot \phi_{J_{\ell}(\mathcal{H}^U)}(K_m^{(\ell)})}{m!} = \frac{\phi(\langle K \rangle)}{2^{\binom{m}{\ell}}},$$

where the second equality follows since $\phi_{\mathcal{H}^U}$ is a coupling of $\phi_{J_{\ell}(\mathcal{H}^U)} \in \text{Hom}^+(\mathcal{A}[T_{\ell}\text{-Hypergraph}], \mathbb{R})$ and $\phi \otimes \psi_{\text{lin}}$ (and the latter satisfies $\text{UInduce}[\ell]$), and the third equality follows from (32). Hence (31) holds.

Let us now show the case $\ell = 1$. In this case, since $T_1\text{-Hypergraph} \cong T_2\text{-Coloring}$, we will work with the latter theory. Let ξ be a coupling of ϕ and $\psi_{1/2} \in \text{Hom}^+(\mathcal{A}[T_2\text{-Coloring}], \mathbb{R})$ and let $\mathcal{N}$ be a $(T \cup T_2\text{-Coloring})$ -on such that $\phi_{\mathcal{N}} = \xi$.

For every $m \in \mathbb{N}$, every $K \in \mathcal{K}_m[T]$ and every $j \in \{0, \dots, m\}$, let $K_j \in \mathcal{K}_m[T \cup T_2\text{-Coloring}]$ be the model obtained from K by coloring the first j vertices with color 1 and all others with color 2, that is, we have $R_P(K_j) \stackrel{\text{def}}{=} R_P(K)$ ($P \in \mathcal{L}$), $R_{\chi_1}(K_j) \stackrel{\text{def}}{=} [j]$ and $R_{\chi_2}(K_j) \stackrel{\text{def}}{=} \{j+1, \dots, m\}$. Again, we let $K_j^{\leq} \in \mathcal{K}_m[T \cup T_2\text{-Coloring} \cup T_{\text{LinOrder}}]$ be the model obtained from K_j by equipping it with the natural order of $[m]$, and, again, in the computations below we view $K, K_j, K_j^{\leq}$ as unlabeled models.

Due to exchangeability, in order to show that ξ is the independent coupling of ϕ and $\psi_{1/2}$, it is sufficient to show that for every $m \in \mathbb{N}$, every $K \in \mathcal{K}_m[T]$ and every $j \in \{0, \dots, m\}$, we have

$$\xi(\langle K_j \rangle) = \frac{\phi(\langle K \rangle)}{2^m}. \quad (33)$$

For every $t \in (0, 1)$, let

$$U_t \stackrel{\text{def}}{=} \{(x, y) \in \mathcal{E}_1 \times \mathcal{E}_1 \mid x \in \mathcal{N}_{\chi_1} \equiv y < t\}$$

(χ_1 corresponds to the first color) and note that $\lambda(U_t) = 1/2$. Define the $(T \cup T_{\text{LinOrder}} \cup T_2\text{-Coloring})$ -on $\mathcal{H}^t$ over $[0, 1]^2$ by

$$\begin{aligned} \mathcal{H}_P^t &\stackrel{\text{def}}{=} \mathcal{N}_P \times \mathcal{E}_{k(P)} & (P \in \mathcal{L}), & & \mathcal{H}_{\chi_1}^t &\stackrel{\text{def}}{=} \mathcal{E}_2 \times \mathcal{N}^{\leq}, \\ \mathcal{H}_{\chi_1}^t &\stackrel{\text{def}}{=} U_t, & & & \mathcal{H}_{\chi_2}^t &\stackrel{\text{def}}{=} (\mathcal{E}_1 \times \mathcal{E}_1) \setminus U_t. \end{aligned}$$

Since $\phi_{\mathcal{H}^t}$ is a coupling of $\psi_{1/2}$ and $\phi \otimes \psi_{\text{lin}}$ and the latter satisfies $\text{UInduce}[1]$, we get

$$\phi_{\mathcal{H}^t}(\langle K_m^{\leq} \rangle) = \frac{\phi(\langle K \rangle)}{m! \cdot 2^m}. \quad (34)$$

On the other hand, from the definition of $\mathcal{H}^t$, we have

$$\begin{aligned}\phi_{\mathcal{H}^t}(\langle K_m^{\leq} \rangle) &= \sum_{j=0}^m \frac{t^j (1-t)^{m-j}}{j!(m-t)!} \xi(\langle K_j \rangle) \\ &= \sum_{k=0}^m \left(\sum_{j=0}^k \frac{1}{j!(m-j)!} \binom{m-j}{k-j} (-1)^{k-j} \xi(\langle K_j \rangle) \right) t^k.\end{aligned}$$

Since this identity is true for any t , putting it together with (34) and comparing coefficients of the polynomials in t , we conclude that

$$\sum_{i=0}^k \frac{1}{i!(m-i)!} \binom{m-i}{k-i} (-1)^{k-i} \xi(\langle K_i \rangle) = \begin{cases} \frac{\phi(\langle K \rangle)}{m! \cdot 2^m}, & \text{if } k = 0; \\ 0, & \text{if } k \in [m]. \end{cases} \quad (35)$$

We can finally prove (33) by induction in $j \in \{0, \dots, m\}$. For $j = 0$, the assertion follows from (35) for $k = 0$. Suppose then that $j \geq 1$ and by using the inductive hypothesis, note that (35) for $k = j$ gives

$$\begin{aligned}\xi(\langle K_j \rangle) &= -j!(m-j)! \sum_{i=0}^{j-1} \frac{1}{i!(m-i)!} \binom{m-i}{j-i} (-1)^{j-i} \frac{\phi(\langle K \rangle)}{2^m} \\ &= - \sum_{i=0}^{j-1} \binom{j}{i} (-1)^{j-i} \frac{\phi(\langle K \rangle)}{2^m} = \frac{\phi(\langle K \rangle)}{2^m}.\end{aligned}$$

Thus (33) holds. □

8 Separations

In this section we prove all separation theorems.

Recall from Section 2.3 that for $x \in \mathcal{E}_n$, $\sigma_x \in S_n$ denotes the unique permutation such that $x_{\{\sigma_x^{-1}(1)\}} < \dots < x_{\{\sigma_x^{-1}(n)\}}$ when the coordinates $(x_{\{i\}} \mid i \in [n])$ are distinct, and is defined arbitrarily otherwise.

Proof of Theorem 3.6. First note that the quasirandom $(\ell + 1)$ -tournamon $\psi_{\ell+1}$ can be represented by the $T_{(\ell+1)}\text{-Tournament-on}$

$$\mathcal{N} \stackrel{\text{def}}{=} \left\{ x \in \mathcal{E}_{\ell+1} \mid x_{[\ell+1]} < \frac{1}{2} \equiv \text{sgn}(\sigma_x) = 1 \right\}. \quad (36)$$

Let $\mathbf{K}$ be the exchangeable array corresponding to $\mathcal{N}$ with respect to $\boldsymbol{\theta}$ picked in $\mathcal{E}_{\mathbb{N}_+}$. By Theorem 3.10, to show that $\psi_{\ell+1} \in \text{UCouple}[\ell]$ it is sufficient to prove that $\psi_{\ell+1}$ is weakly ℓ -independent, that is for every $m \in \mathbb{N}$, the random variable $\mathbf{K}|_{[m]}$ is independent from $(\boldsymbol{\theta}_A \mid A \in r(m, \ell))$. Indeed, $\mathbf{K}|_{[m]}$ is completely determined by $\sigma_{\iota_{[m]}^*}(\boldsymbol{\theta})$ and $(\boldsymbol{\theta}_A \mid A \in \binom{[m]}{\ell+1})$, and any changes in the values of the signs $\text{sgn}(\sigma_{\iota_A^*}(\boldsymbol{\theta}))$ can be offset by flipping the corresponding variables $\boldsymbol{\theta}_A$ (cf. (36)) so that the *distribution* of $\mathbf{K}|_{[m]}$ does not change from fixing $\sigma_{\iota_{[m]}^*}(\boldsymbol{\theta})$.

Suppose now toward a contradiction that $\psi_{\ell+1} \in \text{Independence}[\ell]$, that is $\psi_{\ell+1} = \phi_{\mathcal{H}}$ for some $T_{(\ell+1)\text{-Tournament}}$ -on $\mathcal{H}$ of the form $\mathcal{H} = \mathcal{E}_{\ell+1, \ell} \times \mathcal{G}$ for some $\mathcal{G} \subseteq [0, 1]$. Note that for any $\sigma \in S_{\ell+1}$, we have $\mathcal{H} \cdot \sigma = \mathcal{H}$. But this is a contradiction as the axioms of $T_k\text{-Tournament}$ imply that $\lambda((\mathcal{H} \cdot \sigma) \cap \mathcal{H}) = 0$ whenever $\text{sgn}(\sigma) = -1$. $\square$

Proof of Theorem 3.7. Since ψ_{lin} is represented by the T_{LinOrder} -on $\mathcal{N} \stackrel{\text{def}}{=} \{x \in \mathcal{E}_2 \mid x_{\{1\}} < x_{\{2\}}\}$, we know $\text{rk}(\psi_{\text{lin}}) = 1$, thus by Proposition 4.1, we have $\psi_{\text{lin}} \notin \text{UCouple}[1]$.

Since ψ_{lin} is n -categorical for every $n \in \mathbb{N}$, it is symmetrically ℓ -local for trivial reasons (namely, all events $\mathbf{K}|_{V_i} \cong M_i$ have probability 1), for any integer ℓ . Hence $\psi_{\text{lin}} \in \text{UInduce}[\ell]$ by Theorem 3.11. $\square$

To prove Theorems 3.8 and 3.9, the alternating tournament defined below will play a key role.

Definition 8.1. Let $k \geq 1$. For $\alpha: [k] \rightarrow [k+1]$, denote by σ_α the unique extension of α to an element of S_{k+1} , and let $\text{sgn}(\alpha) \stackrel{\text{def}}{=} \text{sgn}(\sigma_\alpha)$. This definition behaves well with respect to the actions of S_k and S_{k+1} : for every $\eta \in S_k$ we have $\text{sgn}(\alpha \circ \eta) = \text{sgn}(\alpha) \text{sgn}(\eta)$, and for every $\sigma \in S_{k+1}$ we have $\text{sgn}(\sigma \circ \alpha) = \text{sgn}(\sigma) \text{sgn}(\alpha)$.

The *alternating k -tournament* is the model $A_{k+1}^{(k)} \in \mathcal{K}_{k+1}[T_k\text{-Tournament}]$ of $T_k\text{-Tournament}$ of size $k+1$ given by

$$V(A_{k+1}^{(k)}) \stackrel{\text{def}}{=} [k+1]; \quad R_E(A_{k+1}^{(k)}) \stackrel{\text{def}}{=} \{\alpha \in ([k+1])_k \mid \text{sgn}(\alpha) = 1\}.$$

For example, $A_3^{(2)}$ is the oriented cycle $\vec{C}_3$.

Proof of Theorem 3.8. For this proof, let us denote the predicate symbols of $T_{(\ell+2)\text{-Hypergraph}}$ and $T_{(\ell+1)\text{-Tournament}}$ by E and P , respectively. Let $\psi \stackrel{\text{def}}{=}$

$\psi_{\ell+1} \in \text{Hom}^+(\mathcal{A}[T_{(\ell+1)\text{-Tournament}}], \mathbb{R})$ be the quasirandom $(\ell+1)$ -tournament and let $I: T_{(\ell+2)\text{-Hypergraph}} \rightsquigarrow T_{(\ell+1)\text{-Tournament}}$ be given by

$$I(E)(x_1, \dots, x_{\ell+2}) \stackrel{\text{def}}{=} \bigvee_{1 \leq i_1 < \dots < i_\ell \leq \ell+2} (P(x_{i_1}, \dots, x_{i_\ell}, x_{j_1}) \equiv P(x_{i_1}, \dots, x_{i_\ell}, x_{j_2})),$$

where $j_1, j_2 \in [\ell+2]$ are such that $\{i_1, \dots, i_\ell, j_1, j_2\} = [\ell+2]$. By Theorems 3.3 and 3.6, we know that $\phi \stackrel{\text{def}}{=} \psi^I \in \text{Hom}^+(\mathcal{A}[T_{(\ell+2)\text{-Hypergraph}}], \mathbb{R})$ satisfies $\text{UCouple}[\ell]$.

To show that $\phi \notin \text{Independence}[\ell]$, we will make use of the theory T (isomorphic to $T_{(\ell+1)\text{-Tournament}}$) that is obtained from $T_{(\ell+2)\text{-Hypergraph}} \cup T_{(\ell+1)\text{-Tournament}}$ by adding the axiom

$$\forall \vec{x}, E(x_1, \dots, x_{\ell+2}) \equiv I(E)(x_1, \dots, x_{\ell+2}) \quad (37)$$

and the commutative diagram

$$\begin{array}{ccc} T_{(\ell+2)\text{-Hypergraph}} & \xrightarrow{I} & T_{(\ell+1)\text{-Tournament}} \\ S \downarrow & & \downarrow J \\ T_{(\ell+2)\text{-Hypergraph}} \cup T_{(\ell+1)\text{-Tournament}} & \xrightarrow{A} & T \end{array}$$

where S is the structure-erasing interpretation, A is the axiom-adding interpretation and J is the isomorphism mentioned above that acts identically on P (the inverse J^{-1} acts identically on P and acts as I on E). Let $\xi = \psi^{J^{-1}}$ so that $\psi = \xi^J$ and $\phi = \xi^{A \circ S}$.

Suppose toward a contradiction that $\phi \in \text{Independence}[\ell]$ and let $\mathcal{N}$ be an ℓ -independent $T_{(\ell+2)\text{-Hypergraph}}$ -on over Ω such that $\phi_{\mathcal{N}} = \phi = \psi^I$. By Proposition 4.3, there exists a T -on $\mathcal{N}'$ over $\Omega \times \Omega$ such that $\phi_{\mathcal{N}'} = \xi$ and $S(A(\mathcal{N}'))_E = \mathcal{N}'_E = \mathcal{N}_E \times \mathcal{E}_{\ell+2}$ a.e. Note that $\text{rk}(\phi) \leq \text{rk}(\psi) \leq \ell+1$, so by possibly changing zero-measure sets using Proposition 4.2, we may also suppose that $\text{rk}(\mathcal{N}') \leq \ell+1$. By applying a measure-isomorphism between $\Omega \times \Omega$ and $[0, 1]$, we conclude that there exists a T -on $\mathcal{H}$ (over $[0, 1]$) such that $\phi_{\mathcal{H}} = \xi$, $\text{rk}(\mathcal{H}) \leq \ell+1$ and the peon $\mathcal{H}_E$ is ℓ -independent.

Since $\mathcal{H}_E$ has rank at most $\ell+1$ and is ℓ -independent, we can write it as $\mathcal{H}_E = \mathcal{E}_{\ell+2, \ell} \times \mathcal{G} \times [0, 1]^{\{\ell+2\}}$ for some measurable $\mathcal{G} \subseteq [0, 1]^{\binom{\ell+2}{\ell+1}}$. Using the symmetry axiom (4) of $T_{(\ell+2)\text{-Hypergraph}}$ and making a zero-measure change in $\mathcal{G}$, we may assume that it is $S_{\ell+2}$ -invariant.

For every $t \in [\ell + 2]$, define the sets

$$\begin{aligned} V_t^{\ell+1} &\stackrel{\text{def}}{=} \left\{ A \in \binom{[\ell+2]}{t} \mid \ell+1 \in A \wedge \ell+2 \notin A \right\}; \\ V_t^{\ell+2} &\stackrel{\text{def}}{=} \left\{ A \in \binom{[\ell+2]}{t} \mid \ell+1 \notin A \wedge \ell+2 \in A \right\}; \\ V_t^{\ell+1, \ell+2} &\stackrel{\text{def}}{=} \left\{ A \in \binom{[\ell+2]}{t} \mid \ell+1, \ell+2 \in A \right\}. \end{aligned}$$

Define also the sets

$$\begin{aligned} W_t^{\ell+1} &\stackrel{\text{def}}{=} [0, 1]^{V_t^{\ell+1}}; & W_t^{\ell+2} &\stackrel{\text{def}}{=} [0, 1]^{V_t^{\ell+2}}; & W_t^{\ell+1, \ell+2} &\stackrel{\text{def}}{=} [0, 1]^{V_t^{\ell+1, \ell+2}}; \\ Y^{\ell+1} &\stackrel{\text{def}}{=} \prod_{t=1}^{\ell} W_t^{\ell+1}; & Y^{\ell+2} &\stackrel{\text{def}}{=} \prod_{t=1}^{\ell} W_t^{\ell+2}; & Z &\stackrel{\text{def}}{=} \prod_{t=1}^{\ell+2} W_t^{\ell+1, \ell+2}. \end{aligned}$$

Note that

$$\begin{aligned} \mathcal{E}_{\ell+1} &= \mathcal{E}_{\ell} \times Y^{\ell+1} \times W_{\ell+1}^{\ell+1}; \\ \mathcal{E}_{\ell+2} &= \mathcal{E}_{\ell} \times Y^{\ell+1} \times W_{\ell+1}^{\ell+1} \times Y^{\ell+2} \times W_{\ell+1}^{\ell+2} \times Z. \end{aligned}$$

Let $\iota: [\ell] \cup \{\ell+2\} \rightarrow [\ell+1]$ be the function that maps $\ell+2$ to $\ell+1$ and fixes all other points and note that ι induces maps $\iota^*: Y^{\ell+1} \rightarrow Y^{\ell+2}$ and $\iota_{\ell+1}^*: W_{\ell+1}^{\ell+1} \rightarrow W_{\ell+1}^{\ell+2}$ (given by $\iota^*(y)_A \stackrel{\text{def}}{=} y_{\iota(A)}$ and $\iota_{\ell+1}^*(w)_A \stackrel{\text{def}}{=} w_{\iota(A)}$).

For every $x \in \mathcal{E}_{\ell}$ and every $w \in W_{\ell+1}^{\ell+1}$, define the sections

$$\begin{aligned} \mathcal{H}_P^{\alpha}(x, w) &\stackrel{\text{def}}{=} \{y \in Y^{\ell+1} \mid (x, y, w) \in \mathcal{H}_P\}; \\ \mathcal{H}_P^{\beta}(x, w) &\stackrel{\text{def}}{=} \{y \in Y^{\ell+1} \mid (x, y, w) \notin \mathcal{H}_P\}; \end{aligned}$$

and for every $x \in \mathcal{E}_{\ell}$, define

$$\begin{aligned} \mathcal{H}_P^{\alpha}(x) &\stackrel{\text{def}}{=} \{w \in W_{\ell+1}^{\ell+1} \mid \lambda(\mathcal{H}_P^{\alpha}(x, w)) > 0\}; \\ \mathcal{H}_P^{\beta}(x) &\stackrel{\text{def}}{=} \{w \in W_{\ell+1}^{\ell+1} \mid \lambda(\mathcal{H}_P^{\beta}(x, w)) > 0\}. \end{aligned}$$

It is clear that

$$\mathcal{H}_P^{\alpha}(x) \cup \mathcal{H}_P^{\beta}(x) = W_{\ell+1}^{\ell+1} \tag{38}$$

for every $x \in \mathcal{E}_\ell$.

Note that the axiom (37) of T and an application of Fubini's Theorem imply that for a.e. $x \in \mathcal{E}_\ell$, a.e. $w, \widehat{w} \in W_{\ell+1}^{\ell+1}$, a.e. $y \in \mathcal{H}_P^\alpha(x, w)$, a.e. $\widehat{y} \in \mathcal{H}_P^\alpha(x, \widehat{w})$ and a.e. $z \in Z$, we have

$$(x, y, w, \iota^*(\widehat{y}), \iota_{\ell+1}^*(\widehat{w}), z) \in \mathcal{H}_E. \quad (39)$$

Since the definition of $I(P)$ is invariant under negating P , the same assertion also holds with β in place of α .

Recalling that $\mathcal{H}_E = \mathcal{E}_{\ell+2, \ell} \times \mathcal{G} \times [0, 1]^{\{\ell+2\}}$, (39) implies that for a.e. $x \in \mathcal{E}_\ell$, a.e. $w, \widehat{w} \in \mathcal{H}_P^\alpha(x)$ and a.e. $z \in W_{\ell+1}^{\ell+1, \ell+2}$, we have

$$(w, \iota_{\ell+1}^*(\widehat{w}), z) \in \mathcal{G}. \quad (40)$$

Again, the analogous statement with β in place of α also holds.

From (38) and (40), it follows that there exists $x_0 \in \mathcal{E}_\ell$ such that the following hold for $W^\alpha \stackrel{\text{def}}{=} \mathcal{H}_P^\alpha(x_0)$ and $W^\beta \stackrel{\text{def}}{=} \mathcal{H}_P^\beta(x_0)$.

- i. We have $W^\alpha \cup W^\beta = W_{\ell+1}^{\ell+1}$.
- ii. For a.e. $w, \widehat{w} \in W^\alpha$ and a.e. $z \in W_{\ell+1}^{\ell+1, \ell+2}$, we have $(w, \iota_{\ell+1}^*(\widehat{w}), z) \in \mathcal{G}$.
- iii. For a.e. $w, \widehat{w} \in W^\beta$ and a.e. $z \in W_{\ell+1}^{\ell+1, \ell+2}$, we have $(w, \iota_{\ell+1}^*(\widehat{w}), z) \in \mathcal{G}$.

Since $|V_{\ell+1}^{\ell+1}| = 1$, let us for simplicity identify $W_{\ell+1}^{\ell+1}$ with $[0, 1]$ and let $h \stackrel{\text{def}}{=} \mathbb{1}_{W^\alpha}$ be the indicator function of $W^\alpha \subseteq [0, 1]$. For every $A \in \binom{[\ell+2]}{\ell+1}$, let $\pi_A: [0, 1]^{\binom{[\ell+2]}{\ell+1}} \rightarrow [0, 1]$ be the projection on the A -th coordinate and note that the properties above imply that for a.e. $u \in [0, 1]^{\binom{[\ell+2]}{\ell+1}}$, if $h(\pi_{[\ell+1]}(u)) = h(\pi_{[\ell] \cup \{\ell+2\}}(u))$, then $u \in \mathcal{G}$. Since $\mathcal{G}$ is $S_{\ell+2}$ -invariant, this in turn implies that for a.e. $u \in [0, 1]^{\binom{[\ell+2]}{\ell+1}}$, if there exist $j_1, j_2 \in [\ell+2]$ distinct such that $h(\pi_{[\ell+2] \setminus \{j_1\}}(u)) = h(\pi_{[\ell+2] \setminus \{j_2\}}(u))$, then $u \in \mathcal{G}$. But since at least two of the values $h(\pi_{[\ell+1]}(u))$, $h(\pi_{[\ell+2] \setminus \{\ell+1\}}(u))$ and $h(\pi_{[\ell+2] \setminus \{\ell\}}(u))$ must be equal, it follows that $\lambda(\mathcal{G}) = 1$. So we must have

$$\phi(\rho_{\ell+2}) = \lambda(\mathcal{H}_E) = \lambda(\mathcal{G}) = 1,$$

which implies $\phi(\overline{K}_{\ell+2}^{(\ell+2)}) = 0$.

However, note that for the alternating $(\ell + 1)$ -tournament $A_{\ell+2}^{(\ell+1)}$, we have $I(A_{\ell+2}^{(\ell+1)}) \cong \overline{K}_{\ell+2}^{(\ell+2)}$, hence

$$\phi(\overline{K}_{\ell+2}^{(\ell+2)}) \geq \psi(A_{\ell+2}^{(\ell+1)}) = \frac{(\ell + 2)!}{2^{\ell+2} |\text{Aut}(A_{\ell+2}^{(\ell+1)})|} = \frac{1}{2^{\ell+1}},$$

a contradiction. $\square$

The following is needed for the proof of Theorem 3.9.

Lemma 8.2. *If $M \in \mathcal{M}_{k+2}[T_k\text{-Tournament}]$ is a k -tournament on $k + 2$ vertices, then M has at most two (unlabeled) copies of the alternating k -tournament $A_{k+1}^{(k)}$.*

Proof. Suppose toward a contradiction that $M \in \mathcal{M}_{k+2}[T_k\text{-Tournament}]$ contains three copies of $A_{k+1}^{(k)}$ and without loss of generality, let us suppose that these three copies are induced by $V_1 \stackrel{\text{def}}{=} [k + 1]$, $V_2 \stackrel{\text{def}}{=} [k] \cup \{k + 2\}$ and $V_3 \stackrel{\text{def}}{=} [k - 1] \cup \{k + 1, k + 2\}$. Let $\alpha_{12}, \alpha_{13}, \alpha_{23} \in ([k + 2])_k$ be given by

$$\alpha_{12}(v) \stackrel{\text{def}}{=} v; \quad \alpha_{13}(v) \stackrel{\text{def}}{=} \begin{cases} v, & \text{if } v < k; \\ k + 1 & \text{if } v = k; \end{cases} \quad \alpha_{23}(v) \stackrel{\text{def}}{=} \begin{cases} v, & \text{if } v < k; \\ k + 2 & \text{if } v = k; \end{cases}$$

and note that $\text{im}(\alpha_{ij}) = V_i \cap V_j$.

But then $M|_{V_1} \cong A_{k+1}^{(k)}$, $M|_{V_2} \cong A_{k+1}^{(k)}$ and $M|_{V_3} \cong A_{k+1}^{(k)}$ imply respectively that

$$\begin{aligned} \alpha_{12} \in R_E(M) &\equiv \alpha_{13} \notin R_E(M), \\ \alpha_{12} \in R_E(M) &\equiv \alpha_{23} \notin R_E(M), \\ \alpha_{13} \in R_E(M) &\equiv \alpha_{23} \notin R_E(M). \end{aligned}$$

This is a contradiction as all three equivalences above cannot be true at the same time. $\square$

Proof of Theorem 3.9. For this proof, let us again denote the predicate symbols of $T_{(\ell+2)\text{-Hypergraph}}$ and $T_{(\ell+1)\text{-Tournament}}$ by E and P , respectively. For $p \in [0, 1]$, let $\mathcal{N}^p$ be the $T_{(\ell+1)\text{-Tournament-on}}$ given by

$$\mathcal{N}_E^p \stackrel{\text{def}}{=} \{x \in \mathcal{E}_{\ell+1} \mid x_{[\ell+1]} < p \equiv \text{sgn}(\sigma_x) = 1\}$$

(note that for $p = 1/2$ this is precisely the theory (36) representing the quasirandom $(\ell + 1)$ -tournament).

Let $I: T_{(\ell+2)\text{-Hypergraph}} \rightsquigarrow T_{(\ell+1)\text{-Tournament}}$ be the interpretation that declares $(\ell + 2)$ -edges to be isomorphic copies of $A_{\ell+2}^{(\ell+1)}$, and let $\phi_p \stackrel{\text{def}}{=} \phi_{\mathcal{N}^p}^I \in \text{Hom}^+(\mathcal{A}[T_{(\ell+2)\text{-Hypergraph}}], \mathbb{R})$. We will show that ϕ_p satisfies **UInduce** $[\ell]$ for every $p \in [0, 1]$, but does not satisfy **UCouple** $[1]$ unless $p \in \{0, 1/2, 1\}$.

To show the former, recall that the quasirandom $(\ell + 1)$ -hypergraphon $\psi_{\ell+1,p} \in \text{Hom}^+(\mathcal{A}[T_{(\ell+1)\text{-Hypergraph}}], \mathbb{R})$ satisfies **Independence** $[\ell]$ (cf. Lemma 7.4) and hence **UCouple** $[\ell]$ (by Theorem 3.2). Note also that $\phi_{\mathcal{N}^p} = (\psi_{\ell+1,p} \otimes \psi_{\text{lin}})^{I'}$ and $I': T_{(\ell+1)\text{-Tournament}} \rightsquigarrow T_{(\ell+1)\text{-Hypergraph}} \cup T_{\text{LinOrder}}$ is given by⁷

$$I'(P)(x_1, \dots, x_{\ell+1}) \stackrel{\text{def}}{=} \left(\bigwedge_{1 \leq i < j \leq \ell+1} x_i \neq x_j \right) \wedge \left(E(x_1, \dots, x_{\ell+1}) \equiv \bigvee_{\substack{\sigma \in S_{\ell+1} \\ \text{sgn}(\sigma)=1}} \bigwedge_{1 \leq i < j \leq \ell+1} x_{\sigma(i)} \prec x_{\sigma(j)} \right).$$

By Theorem 3.10(i) $\implies$ (vii), we know that $\psi_{\ell+1,p} \otimes \psi_{\text{lin}} \in \text{UInduce}[\ell]$ and by Theorem 3.3, we get that $\phi_p = (\psi_{\ell+1,p} \otimes \psi_{\text{lin}})^{I' \circ I}$ satisfies **UInduce** $[\ell]$.

Let us now show that for every $p \in (0, 1) \setminus \{1/2\}$, ϕ_p does not satisfy **UCouple** $[1]$. Since ψ_{lin} has rank 1, it is enough to show that ϕ_p is not uniquely coupleable with ψ_{lin} . Consider the $(T_{(\ell+1)\text{-Tournament}} \cup T_{\text{LinOrder}})$ -on $\mathcal{N}^{p,<}$ given by

$$\mathcal{N}_P^{p,<} \stackrel{\text{def}}{=} \mathcal{N}_P^p; \quad \mathcal{N}_{\prec}^{p,<} \stackrel{\text{def}}{=} \{x \in \mathcal{E}_2 \mid x_{\{1\}} < x_{\{2\}}\}$$

and note that $\phi_{\mathcal{N}^{p,<}}$ is a coupling of $\phi_{\mathcal{N}^p}$ and ψ_{lin} , hence $\xi \stackrel{\text{def}}{=} \phi_{\mathcal{N}^{p,<}}^{I \cup \text{Id}_{T_{\text{LinOrder}}}}$ is a coupling of ϕ_p and ψ_{lin} . We will show that $\xi \neq \phi_p \otimes \psi_{\text{lin}}$ by a direct computation exhibiting an $(\ell + 2)$ -hypergraph H and two different orders on it such that $\xi(H_1) \neq \xi(H_2)$ for the corresponding models of the theory $T_{(\ell+2)\text{-Hypergraph}} \cup T_{\text{LinOrder}}$. That will suffice since, clearly, $(\phi_p \otimes \psi_{\text{lin}})(H_1) = (\phi_p \otimes \psi_{\text{lin}})(H_2)$.

Let $H \in \mathcal{K}_{\ell+3}[T_{(\ell+2)\text{-Hypergraph}}]$ be given by

$$V(H) \stackrel{\text{def}}{=} [\ell + 3]; \quad E(H) \stackrel{\text{def}}{=} \{[k + 1], [k] \cup [k + 2]\};$$

⁷This is a generalization of the “arc-orientation” interpretation used implicitly in the implications $P_{10} \implies P_{11} \implies P_1(s)$ of [CG91].

and let $H_1, H_2 \in \mathcal{K}_{\ell+3}[T_{(\ell+2)\text{-Hypergraph}} \cup T_{\text{LinOrder}}]$ be obtained from H by equipping it with the orders $\prec_1$ and $\prec_2$, respectively, where $\prec_1$ is the natural order of $[\ell+3]$ and $\prec_2$ is obtained from $\prec_1$ by swapping the order position of $\ell+1$ and $\ell+3$, that is, we have

$$1 \prec_2 2 \prec_2 \cdots \prec_2 \ell \prec_2 \ell+3 \prec_2 \ell+2 \prec_2 \ell+1.$$

Let θ be picked at random in $\mathcal{E}_{\mathbb{N}_+}$ according to λ and let $\mathbf{K}$ be the exchangeable array corresponding to $\mathcal{N}^{p,<}$ with respect to θ (so that $(I \cup \text{id}_{T_{\text{LinOrder}}})(\mathbf{K})$ corresponds to $(I \cup \text{id}_{T_{\text{LinOrder}}})(\mathcal{N}^{p,<})$). Let $\sigma \stackrel{\text{def}}{=} \sigma_{\iota_{[\ell+3]}^*}(\theta)$. Then we have

$$\begin{aligned} \xi(\langle H_1 \rangle) &= \mathbb{P}[I(J(\mathbf{K}|_{[\ell+3]})) = H \wedge \sigma = \text{id}_{\ell+3}]; \\ \xi(\langle H_2 \rangle) &= \mathbb{P}[I(J(\mathbf{K}|_{[\ell+3]})) = H \wedge \sigma = \tau]; \end{aligned}$$

where $J: T_{(\ell+1)\text{-Tournament}} \rightsquigarrow T_{(\ell+1)\text{-Tournament}} \cup T_{\text{LinOrder}}$ is the structure-erasing interpretation and τ is the transposition that swaps $\ell+1$ and $\ell+3$. Then by Lemma 8.2, $I(J(\mathbf{K}|_{[\ell+3]})) = H$ is equivalent to

$$J(\mathbf{K}|_{[\ell+2]}) \cong J(\mathbf{K}|_{[\ell+1] \cup \{\ell+3\}}) \cong A_{\ell+2}^{(\ell+1)}. \quad (41)$$

Since $\text{Aut}(A_{\ell+2}^{(\ell+1)})$ is the alternating group on $[\ell+2]$, on any fixed set of $\ell+2$ vertices, there are exactly two models M_1 and M_2 that are isomorphic to $A_{\ell+2}^{(\ell+1)}$ and they satisfy $R_P(M_1) \cap R_P(M_2) = \emptyset$. This means that on the event (41), out of the a priori four presentations of $A_{\ell+2}^{(\ell+1)}$ induced on $[\ell+2]$ and $[\ell+1] \cup \{\ell+3\}$, only two are actually possible. Since ℓ is odd, a straightforward calculation gives

$$\begin{aligned} \xi(\langle H_1 \rangle) &= p^{(\ell+2)}(1-p)^{\ell+1} + p^{\ell+1}(1-p)^{\ell+2} = p^{\ell+1}(1-p)^{\ell+1}; \\ \xi(\langle H_2 \rangle) &= p^\ell(1-p)^{\ell+3} + p^{\ell+3}(1-p)^\ell = p^\ell(1-p)^\ell(3p^2 - 3p + 1). \end{aligned}$$

Thus we get

$$\begin{aligned} \xi(\langle H_2 \rangle) - \xi(\langle H_1 \rangle) &= p^\ell(1-p)^\ell(4p^2 - 4p + 1) \\ &= p^\ell(1-p)^\ell(2p-1)^2, \end{aligned}$$

which is non-zero as long as $p \in (0, 1) \setminus \{1/2\}$. $\square$

Proof of Theorem 3.15. For $p \in (0, 1)$, let $\mathcal{N}$ be the T_k -Hypergraph-on given by

$$\mathcal{N} \stackrel{\text{def}}{=} \left\{ x \in \mathcal{E}_k \mid \left(\min\{x_{\{v\}} \mid v \in [k]\} < 1/2 \wedge x_{[k]} < p \right) \right. \\ \left. \vee \left(\min\{x_{\{v\}} \mid v \in [k]\} \geq 1/2 \wedge \sum_{v \in [k]} x_{[k] \setminus \{v\}} \bmod 1 < p \right) \right\}.$$

Let us show that $\phi \stackrel{\text{def}}{=} \phi_{\mathcal{N}}$ satisfies $\text{Dev}[k-1]$; recall that $\text{Dev}[k-1] = \text{Disc}[\mathcal{A}_{k-1}]$, where $\mathcal{A}_{k-1} \stackrel{\text{def}}{=} \{A \in \binom{[k]}{k-1} \mid \{1\} \subseteq A\} = \binom{[k]}{k-1} \setminus \{[k] \setminus \{1\}\}$ (see Definition 2.8) and for $\psi \in \text{Hom}^+(\mathcal{A}[T_{\mathcal{L}_{\mathcal{A}_{k-1}}}], \mathbb{R})$, let ξ be a coupling of ϕ and ψ . By Proposition 4.3, there exists a $(T \cup T_{\mathcal{L}_{\mathcal{A}_{k-1}}})$ -on $\mathcal{H}$ over $[0, 1]^2$ such that $\phi_{\mathcal{H}} = \xi$ and $\mathcal{H}_E = \mathcal{N} \times \mathcal{E}_k$.

Let (θ^1, θ^2) be picked in $\mathcal{E}_{\mathbb{N}_+}([0, 1]^2)$ according to λ and let $\mathbf{K}$ be the exchangeable array corresponding to $\mathcal{H}$ with respect to (θ^1, θ^2) . Our objective is to show that the events $(1, 2, \dots, k) \in R_E(\mathbf{K})$ and $\forall A \in \mathcal{A}_{k-1}, \iota_A \in R_{P_A}(\mathbf{K})$ are independent.

Since the event $\iota_A \in R_{P_A}(\mathbf{K})$ is completely determined by $((\theta_B^1, \theta_B^2) \mid B \subseteq A)$, it is sufficient to show that the event $(1, \dots, k) \in R_E(\mathbf{K})$ is independent from $((\theta_B^1, \theta_B^2) \mid B \in r(k, k-1) \wedge B \neq [k] \setminus \{1\})$. But the event $(1, \dots, k) \in R_E(\mathbf{K})$ is equivalent to $(\theta_B^1)_{B \in r(k)} \in \mathcal{N}$, and it is easy to see that the conditional probability of $(1, \dots, k) \in R_E(\mathbf{K})$ given $((\theta_B^1, \theta_B^2) \mid B \in r(k, k-1) \wedge B \neq [k] \setminus \{1\})$ is p a.e. Hence ϕ satisfies $\text{Dev}[k-1]$.

Let us now show that ϕ does not satisfy $\text{UInduce}[1]$. To do so, for each $i \in [2]$, we consider the $(T_k\text{-Hypergraph} \cup T_2\text{-Coloring})$ -on $\mathcal{H}^i$ (see Remark 1) given by

$$\begin{aligned} \mathcal{H}_E &= \mathcal{N}; \\ \mathcal{H}_{\chi_i} &= \{x \in \mathcal{E}_1 \mid x_{\{1\}} < 1/2\}; \\ \mathcal{H}_{\chi_{3-i}} &= \{x \in \mathcal{E}_1 \mid x_{\{1\}} \geq 1/2\}. \end{aligned}$$

Then by a straightforward calculation, for every $H \in \mathcal{M}[T_k\text{-Hypergraph} \cup T_2\text{-Coloring}]$ with $R_{\chi_1}(H) = V(H)$, we have

$$\phi_{\mathcal{H}^1}(H) = \frac{\psi_{k,p}(I(H))}{2^{|H|}}; \quad \phi_{\mathcal{H}^2}(H) = \frac{\phi_{\mathcal{N}'}(I(H))}{2^{|H|}};$$

where $I: T_k\text{-Hypergraph} \rightsquigarrow T_k\text{-Hypergraph} \cup T_2\text{-Coloring}$ is the structure-erasing interpretation, $\psi_{k,p}$ is the quasirandom k -hypergraphon (see Definition 2.11)

and $\mathcal{N}'$ is the T_k -Hypergraph-on given by

$$\mathcal{N}' = \left\{ x \in \mathcal{E}_k \left| \sum_{v \in [k]} x_{[k] \setminus \{v\}} \bmod 1 < p \right. \right\}.$$

Since $\phi_{\mathcal{N}'} \neq \psi_{k,p}$ (since $\text{rk}(\psi_{k,p}) = k > k-1 \geq \text{rk}(\psi_{\mathcal{N}'})$), it follows that $\phi_{\mathcal{H}^1}(H) \neq \phi_{\mathcal{H}^2}(H)$ for some $H \in \mathcal{M}[T_k\text{-Hypergraph} \cup T_2\text{-Coloring}]$ with $R_{\chi_1}(H) = V(H)$, hence ϕ does not satisfy **UInduce**[1]. $\square$

Proof of Theorem 3.16. For $p \in (0, 1)$, let $\mathcal{N}$ be the T_k -Hypergraph-on given by

$$\mathcal{N} \stackrel{\text{def}}{=} \left\{ x \in \mathcal{E}_k \left| \max \left\{ x_A \mid A \in \binom{[k]}{\ell+1} \right\} < p \right. \right\}.$$

It is clear that $\phi \stackrel{\text{def}}{=} \phi_{\mathcal{N}}$ satisfies **Independence**[\(\ell\)]. Consider now the $T_{\mathcal{L}_{\{[\ell+1]\}}}$ -on $\mathcal{H}$ given by

$$\mathcal{H}_E \stackrel{\text{def}}{=} \mathcal{N}; \quad \mathcal{H}_{P_{[\ell+1]}} \stackrel{\text{def}}{=} \{x \in \mathcal{E}_{\ell+1} \mid x_{[\ell+1]} \geq p\}$$

and note that if $\mathbf{K}$ is the exchangeable array corresponding to $\mathcal{H}$, then

$$\begin{aligned} \mathbb{P}[(1, \dots, k) \in R_E(\mathbf{K}) \wedge (1, \dots, \ell+1) \in R_{P_{[\ell+1]}}(\mathbf{K})] &= 0 \\ &\neq p^{\binom{k}{\ell+1}} \cdot (1-p) = \phi(\rho_k) \cdot \mathbb{P}[(1, \dots, \ell+1) \in R_{P_{[\ell+1]}}(\mathbf{K})], \end{aligned}$$

so ϕ does not satisfy **Disc**[\(\{[\ell+1]\}\)]. $\square$

Proof of Theorem 3.5. Follows from Theorems 3.14 (**UInduce**[\(\ell+1\)] $\implies$ **CliqueDisc**[\(\ell+1\)]) and 3.16 (**Independence**[\(\ell\)] $\not\Rightarrow$ **Disc**[\(\{[\ell+1]\}\)]), and the fact that **CliqueDisc**[\(\ell+1\)] $\implies$ **Disc**[\(\{[\ell+1]\}\)] (see [Tow17, AHCH⁺18]). $\square$

9 Top level quasirandomness

In this section we prove Theorems 3.12 and 3.13, which completely characterize the properties **Independence**[\(k-1\)] and **UCouple**[\(k-1\)], respectively when all arities are at most k . These can be seen as analogues of full quasirandomness for arbitrary universal theories (just as $\text{Dev}[k] = \text{CliqueDisc}[k-1] = \text{Disc}[\binom{[k]}{k-1}]$ gives full quasirandomness in T_k -Hypergraph).

Proof of Theorem 3.12. By Lemma 7.4, $\psi_{k,p} \in \text{Hom}^+(\mathcal{A}[T_{c,k}], \mathbb{R})$ satisfies **Independence** $[k-1]$, so the backward direction follows from Theorem 3.3.

For the forward direction, first we claim that it is enough to show the case when $T = T_{\mathcal{L}}$. (This is not completely immediate as $I: T \rightsquigarrow T_{c,k}$ is required to satisfy $T_{c,k} \vdash \forall \vec{x}, I(F)(\vec{x})$ for every axiom $\forall \vec{x}, F(\vec{x})$ of T .) Let $A: T_{\mathcal{L}} \rightsquigarrow T$ be the axiom-adding interpretation and suppose ϕ^A (which satisfies **UCouple** $[k-1]$ by Theorem 3.3) can be written as $\phi^A = \psi_{k,p}^J$ for some $c \geq 2$, some $p \in \Pi_c$ and some $J: T_{\mathcal{L}} \rightsquigarrow T_{c,k}$, then we define $I: T \rightsquigarrow T_{c,k}$ to act as J and we have to show that it is indeed an interpretation, i.e., that $T_{c,k} \vdash \forall \vec{x}, I(F)(\vec{x})$ for every axiom $\forall \vec{x}, F(\vec{x})$ of T ($\psi_{k,p}^I = \phi$ will then follow trivially). Equivalently, we have to show that if $M \in \mathcal{M}[T_{c,k}]$, then $J(M) \in \mathcal{M}[T]$. But since all p_i are positive, we have $\psi_{k,p}(M) > 0$, so $\phi^A(J(M)) > 0$, hence trivially $J(M) \in \mathcal{M}[T]$.

Let us now prove the case $T = T_{\mathcal{L}}$. Let $\mathcal{N}$ be a $(k-1)$ -independent $T_{\mathcal{L}}$ -on such that $\phi_{\mathcal{N}} = \phi$. Note that if $P \in \mathcal{L}$ is such that $k(P) \leq k-1$, then $\mathcal{N}_P$ must be either $\emptyset$ or $\mathcal{E}_{k(P)}$, so we can write $\mathcal{L} = \mathcal{L}' \cup \mathcal{L}_0 \cup \mathcal{L}_1$, where

$$\begin{aligned}\mathcal{L}' &\stackrel{\text{def}}{=} \{P \in \mathcal{L} \mid k(P) = k\}; \\ \mathcal{L}_0 &\stackrel{\text{def}}{=} \{P \in \mathcal{L} \mid k(P) \leq k-1 \wedge \mathcal{N}_P = \emptyset\}; \\ \mathcal{L}_1 &\stackrel{\text{def}}{=} \{P \in \mathcal{L} \mid k(P) \leq k-1 \wedge \mathcal{N}_P = \mathcal{E}_{k(P)}\}.\end{aligned}$$

Recall from Definition 6.8 that $\mathcal{K}_k[\text{Th}(\phi)] = \{K \in \mathcal{K}_k[T_{\mathcal{L}}] \mid \phi(K) > 0\}$ and enumerate its elements as $K_1, \dots, K_c$. Note that since $\mathcal{N}$ is $(k-1)$ -independent, it follows that every peon $\mathcal{N}_P$ with $P \in \mathcal{L}'$ is S_k -invariant, hence we must have $\text{Aut}(K_i) = S_k$ for every $i \in [c]$. Suppose first that $c \geq 2$ and define $p \in \Pi_c$ by $p_i = \phi(K_i) > 0$ and let $I: T_{\mathcal{L}} \rightsquigarrow T_{c,k}$ be given by

$$\begin{aligned}I(P)(x_1, \dots, x_{k(P)}) &\stackrel{\text{def}}{=} x_1 \neq x_1 && (P \in \mathcal{L}_0); \\ I(P)(x_1, \dots, x_{k(P)}) &\stackrel{\text{def}}{=} \bigwedge_{1 \leq i < j \leq k(P)} x_i \neq x_j && (P \in \mathcal{L}_1); \\ I(P)(x_1, \dots, x_k) &\stackrel{\text{def}}{=} \bigvee_{\substack{i \in [c] \\ \text{id}_k \in R_P(K_i)}} E_i(x_1, \dots, x_k). && (P \in \mathcal{L}').\end{aligned} \quad (42)$$

Since $\mathcal{N}$ is $(k-1)$ -independent, it follows that each $T_{\text{ind}}(K_i, \mathcal{N})$ is $(k-1)$ -independent and has measure p_i , which implies that the $T_{c,k}$ -on $\mathcal{H}$ defined by

$\mathcal{H}_{E_i} \stackrel{\text{def}}{=} T_{\text{ind}}(K_i, \mathcal{N})$ ($i \in [c]$) satisfies $\phi_{\mathcal{H}} = \psi_{k,p}$ and since clearly $I(\mathcal{H}) = \mathcal{N}$, it follows that $\psi_{k,p}^I = \phi$.

If $c = 1$, then we can define I by replacing (42) with

$$\begin{aligned} I(P)(x_1, \dots, x_k) &\stackrel{\text{def}}{=} \bigwedge_{1 \leq i < j \leq k(P)} x_i \neq x_j & (P \in \mathcal{L}', \text{id}_k \in R_P(K_1)); \\ I(P)(x_1, \dots, x_k) &\stackrel{\text{def}}{=} x_1 \neq x_1 & (P \in \mathcal{L}', \text{id}_k \notin R_P(K_1)) \end{aligned}$$

instead and we trivially get $\phi = \psi_{k,p}^I$ for any $p \in \Pi_{c'}$ with $c' \geq 2$ as we must have $T_{\text{ind}}(K_1, \mathcal{N}) = \mathcal{E}_k$ a.e. $\square$

Before we show Theorem 3.13, let us first see that the (Θ, p) -quasirandom homomorphisms $\psi_{\Theta,p} \in \text{Hom}^+(\mathcal{A}[T_{\Theta}], \mathbb{R})$ from Definition 2.9 are well-defined (i.e., their definition as $\psi_{\Theta,p} \stackrel{\text{def}}{=} \phi_{\mathcal{N}^Z}$ is independent of the choice of Z) and satisfy $\text{UCouple}[k-1]$.

Proposition 9.1. *With the notation and conditions of Definition 2.9, we have*

$$\phi_{\mathcal{N}^Z}(\langle M \rangle) = \prod_{P \in \mathcal{L}} p_P^{|R_P(M)|/k!} \quad (43)$$

for every $M \in \mathcal{M}[T_{\Theta}]$. Furthermore, $\psi_{\Theta,p} \stackrel{\text{def}}{=} \phi_{\mathcal{N}^Z}$ satisfies $\text{UCouple}[k-1]$.

Proof. First, let us show that $\mathcal{N}^Z$ is indeed a T_{Θ} -on.

Note first that T_{Θ} trivially proves that

$$\neg P(x, y, \dots, t) \quad (P \in \mathcal{L}, \text{the tuple } (x, y, \dots, t) \text{ contains repeated variables}) \quad (44)$$

and if we add (44) to the axioms of T_{Θ} , then it becomes substitutionally closed (see [CR20, Definition 3.5, Remark 5]), then by [CR20, Theorem 3.7], to show that $\mathcal{N}^Z$ is a T_{Θ} -on, it is enough to show that $\mathcal{N}^Z$ satisfies the axioms of T_{Θ} and (44) a.e. It is trivial that $\mathcal{N}^Z$ satisfies (44) a.e.

Note that the fact that Z is a partition implies that there exists a unique $P_x \in \mathcal{L}$ such that $x_{[k]} \in Z_{P_x}$, thus there exists a unique $Q_x \in \mathcal{L}$ such that $x \in \mathcal{N}_{Q_x}^Z$, namely $Q_x = \sigma_x^{-1} \cdot P_x$ (where σ_x is as in Definition 2.9). This implies that $\mathcal{N}^Z$ satisfies axioms (8) and (10) a.e.

Note now that if $\tau \in S_k$, then we have $\sigma_{x \cdot \tau} = \sigma_x \circ \tau$, hence

$$x \cdot \tau \in \mathcal{N}_P^Z \equiv x_{[k]} \in Z_{\sigma_{x \cdot \tau} \cdot P} \equiv x_{[k]} \in Z_{\sigma_x \cdot (\tau \cdot P)} \equiv x \in \mathcal{N}_{\tau \cdot P}^Z,$$

so $\mathcal{N}^Z$ also satisfies axiom (9) a.e., hence $\mathcal{N}^Z$ is a T_Θ -on.

Let $\mathbf{K}$ be the exchangeable array corresponding to $\mathcal{N}^Z$ with respect to $\boldsymbol{\theta}$ picked in $\mathcal{E}_{\mathbb{N}_+}$ according to λ . Since for $m \in \mathbb{N}$ and $K \in \mathcal{K}_m[T_\Theta]$, we have $\phi_{\mathcal{N}^Z}(\langle K \rangle) = \mathbb{P}[\mathbf{K}|_{[m]} = K]$, if we show that for every measurable $U \subseteq \mathcal{E}_{m,k-1}$ with $\lambda(U) > 0$, we have

$$\mathbb{P}[\mathbf{K}|_{[m]} = K \mid E] = \prod_{P \in \mathcal{L}} p_P^{|R_P(K)|/k!}, \quad (45)$$

where E is the event $(\boldsymbol{\theta}_B \mid B \in r(m, k-1)) \in U$, then both (43) and $\psi_{\Theta, p} \in \text{UCouple}[k-1]$ will follow (the former follows by taking $U = \mathcal{E}_{m,k-1}$ and the latter implies weak $(k-1)$ -independence of $\mathcal{N}^Z$, which is equivalent to $\phi_{\mathcal{N}^Z} \in \text{UCouple}[k-1]$ by Theorem 3.10).

If $m < k$, (45) trivially holds, so suppose $m \geq k$ and note that the axioms of T_Θ imply that for each $\alpha: [k] \rightarrow [m]$, there exists a unique $P_\alpha \in \mathcal{L}$ such that $\alpha \in R_{P_\alpha}(K)$ and we must further have $P_\alpha = \tau \cdot P_{\alpha \circ \tau}$ for every $\tau \in S_k$. Note that for any choice of $(\alpha_A)_{A \in \binom{[m]}{k}}$ with $\alpha_A: [k] \rightarrow [m]$ and $\text{im}(\alpha_A) = A$, we have

$$\begin{aligned} \mathbb{P}[\mathbf{K}|_{[m]} = K \mid E] &= \mathbb{P}[\forall \alpha \in ([m])_k, \alpha \in R_{P_\alpha}(\mathbf{K}) \mid E] \\ &= \mathbb{P}\left[\forall A \in \binom{[m]}{k}, \alpha_A \in R_{P_{\alpha_A}}(\mathbf{K}) \mid E\right]. \end{aligned}$$

Now, the event $\alpha_A \in R_{P_{\alpha_A}}(\mathbf{K})$ depends only on the relative order of $(\boldsymbol{\theta}_{\{i\}} \mid i \in A)$ and on the variable $\boldsymbol{\theta}_A$ and, since p is Θ -invariant, we have $\lambda(Z_{\sigma \cdot P_\alpha}) = p_{P_\alpha}$ for every $\sigma \in S_k$ and every $\alpha: [k] \rightarrow [m]$. This means that if $\leq$ is an ordering of A and $E_{\leq}$ is the event that says that the relative order of $(\boldsymbol{\theta}_{\{i\}} \mid i \in A)$ is $\leq$, then $\mathbb{P}[\alpha \in R_{P_\alpha}(\mathbf{K}) \mid E \wedge E_{\leq}] = p_{P_\alpha}$ and thus

$$\mathbb{P}[\mathbf{K}|_{[m]} = K \mid E] = \prod_{A \in \binom{[m]}{k}} p_{P_{\alpha_A}}.$$

Since this holds for any choice of $(\alpha_A)_{A \in \binom{[m]}{k}}$ with $\text{im}(\alpha_A) = A$, by considering all possible $k! \binom{m}{k}$ such choices we get

$$\mathbb{P}[\mathbf{K}|_{[m]} = K \mid E]^{k! \binom{m}{k}} = \prod_{P \in \mathcal{L}} p_P^{k! \binom{m}{k}^{-1} \cdot |R_P(K)|},$$

from which (45) follows. $\square$

Definition 9.2. Given a T -on $\mathcal{N}$ over $\Omega = (X, \mathcal{A}, \mu)$ and $K \in \mathcal{K}_V[T]$, let $W_{\mathcal{N}}^K: \mathcal{E}_{V,|V|-1}(\Omega) \rightarrow [0, 1]$ be defined by

$$W_{\mathcal{N}}^K(x) \stackrel{\text{def}}{=} \mu(\{y \in X \mid (x, y) \in T_{\text{ind}}(K, \mathcal{N})\}).$$

Note that $W_{\mathcal{N}}^K$ is essentially a $(|V|-1)$ -flattening of the peon $T_{\text{ind}}(K, \mathcal{N}) \subseteq \mathcal{E}_V(\Omega)$ (see Definition 6.4).

The next two simple lemmas are fundamental in the proof of Theorem 3.13.

Lemma 9.3. *Let $k \in \mathbb{N}_+$ and suppose that $k(P) \leq k$ for all $P \in \mathcal{L}$. Let T be a theory over $\mathcal{L}$ and $\mathcal{N}$ be a T -on over $\Omega = (X, \mathcal{A}, \mu)$. Then for every $m \in \mathbb{N}$ and every $K \in \mathcal{K}_m[T]$, we have*

$$\phi_{\mathcal{N}}(\langle K \rangle) = \int_{X^{r(m, k-1)}} \prod_{A \in \binom{[m]}{k}} W_{\mathcal{N}}^{K|_A}(\pi_A(x)) \, d\mu(x),$$

where $\pi_A: \mathcal{E}_{m, k-1}(\Omega) \rightarrow \mathcal{E}_{A, k-1}(\Omega)$ is the projection on the coordinates indexed by $r(A, k-1)$.

Proof. Follows by considering the exchangeable array corresponding to $\mathcal{N}$ with respect to θ picked in $\mathcal{E}_{\mathbb{N}_+}(\Omega)$ according to μ , noting that $\mathbf{K}|_{[m]} = K$ is equivalent to $\forall A \in \binom{[m]}{k}, \mathbf{K}|_A = K|_A$ (since $k(P) \leq k$ for every $P \in \mathcal{L}$) and integrating out the top variables $(\theta_A \mid A \in \binom{[m]}{k})$. $\square$

Lemma 9.4. *If a T -on $\mathcal{N}$ over Ω is such that $\phi_{\mathcal{N}}$ satisfies $\text{UCouple}[\ell]$ and $K \in \mathcal{K}_V[T]$ with $|V| \leq \ell + 1$, then $W_{\mathcal{N}}^K$ is a.e. constant.*

Proof. Without loss of generality, we may suppose that $V = [m]$. Write $\Omega = (X, \mathcal{A}, \mu)$. Then it is sufficient to show that for every measurable $U \in \mathcal{E}_{m, \ell}(\Omega)$, we have $\int_U W_{\mathcal{N}}^K \, d\mu = \mu(U) \phi_{\mathcal{N}}(\langle K \rangle)$. But for the exchangeable array $\mathbf{K}$ corresponding to $\mathcal{N}$ with respect to θ picked in $\mathcal{E}_{\mathbb{N}_+}(\Omega)$ according to μ , it follows that

$$\begin{aligned} \int_U W_{\mathcal{N}}^K \, d\mu &= \mathbb{P}[\mathbf{K}|_{[m]} = K \wedge (\theta_A \mid A \in r(m, k-1)) \in U] \\ &= \mathbb{P}[\mathbf{K}|_{[m]} = K] \cdot \mathbb{P}[(\theta_A \mid A \in r(m, k-1)) \in U] = \mu(U) \phi_{\mathcal{N}}(\langle K \rangle), \end{aligned}$$

where the second equality follows since $\mathcal{N}$ is weakly ℓ -independent by Theorem 3.10. $\square$

Proof of Theorem 3.13. The backward direction follows from Proposition 9.1 and Theorem 3.3.

For the forward direction, we will show that in fact we can take $p = (p_P)_{P \in \mathcal{L}}$ satisfying $p_P > 0$ for every $P \in \mathcal{L}$. Note that when $p_P > 0$ for every $P \in \mathcal{L}$, we have $\psi_{\Theta, p}(M) > 0$ for every $M \in \mathcal{M}[T_\Theta]$, so by an argument analogous to that of the proof of Theorem 3.12, it is enough to consider the case when $T = T_{\mathcal{L}}$.

Suppose then that $T = T_{\mathcal{L}}$ and let $\mathcal{N}$ be a T -on such that $\phi_{\mathcal{N}} = \phi$. Note that if $P \in \mathcal{L}$ is such that $k(P) \leq k - 1$, then $\text{rk}(\mathcal{N}_P) \leq k - 1$, so by Theorem 3.3 and Proposition 4.1, it follows that $\text{rk}(\mathcal{N}_P) = 0$, that is, $\lambda(\mathcal{N}_P) \in \{0, 1\}$. This means that we can write $\mathcal{L} = \widehat{\mathcal{L}} \cup \mathcal{L}_0 \cup \mathcal{L}_1$, where

$$\begin{aligned}\widehat{\mathcal{L}} &\stackrel{\text{def}}{=} \{P \in \mathcal{L} \mid k(P) = k\}; \\ \mathcal{L}_i &\stackrel{\text{def}}{=} \{P \in \mathcal{L} \mid k(P) \leq k - 1 \wedge \lambda(\mathcal{N}_P) = i\} \quad (i \in \{0, 1\}).\end{aligned}$$

Consider the (left) action of S_k on $\mathcal{K}_k[\text{Th}(\phi)]$ given by letting $\sigma \cdot K \in \mathcal{K}_k[\text{Th}(\phi)]$ ($\sigma \in S_k$, $K \in \mathcal{K}_k[\text{Th}(\phi)]$) be the model obtained from K by permuting its vertices by σ , that is, we have

$$\begin{aligned}R_P(\sigma \cdot K) &\stackrel{\text{def}}{=} \{\sigma \circ \alpha \mid \alpha \in R_P(K)\} && (P \in \widehat{\mathcal{L}}); \\ R_P(\sigma \cdot K) &\stackrel{\text{def}}{=} \emptyset && (P \in \mathcal{L}_0); \\ R_P(\sigma \cdot K) &\stackrel{\text{def}}{=} ([k])_{k(P)} && (P \in \mathcal{L}_1).\end{aligned}$$

Note that this definition ensures that for a.e. $x \in \mathcal{E}_k$ and every $\sigma \in S_k$, we have

$$x \cdot \sigma \in T_{\text{ind}}(K, \mathcal{N}) \equiv x \in T_{\text{ind}}(\sigma \cdot K, \mathcal{N}). \quad (46)$$

It is also clear that for a.e. $x \in \mathcal{E}_k$, there exists exactly one $K \in \mathcal{K}_k[\text{Th}(\phi)]$ such that $x \in T_{\text{ind}}(K, \mathcal{N})$.

Let then $\mathcal{L}'$ be a language containing one predicate symbol P_K of arity k for each $K \in \mathcal{K}_k[\text{Th}(\phi)]$ and let $\Theta: S_k \times \mathcal{L}' \rightarrow \mathcal{L}'$ be the induced action $\sigma \cdot P_K \stackrel{\text{def}}{=} P_{\sigma \cdot K}$ ($\sigma \in S_k$, $K \in \mathcal{K}_k[\text{Th}(\phi)]$). Define then $\mathcal{H}$ by

$$\mathcal{H}_{P_K} \stackrel{\text{def}}{=} T_{\text{ind}}(K, \mathcal{N})$$

and note that (46) and the remark below it ensure that $\mathcal{H}$ is a T_Θ -on.

Define $I: T \rightsquigarrow T_\Theta$ by

$$I(P)(x_1, \dots, x_{k(P)}) \stackrel{\text{def}}{=} \begin{cases} \bigvee_{\substack{K \in \mathcal{K}_k[\text{Th}(\phi)] \\ \text{id}_k \in R_P(K)}} P_K(x_1, \dots, x_{k(P)}), & \text{if } P \in \widehat{\mathcal{L}}; \\ x_1 \neq x_1, & \text{if } P \in \mathcal{L}_0; \\ \bigwedge_{1 \leq i < j \leq k(P)} x_i \neq x_j, & \text{if } P \in \mathcal{L}_1. \end{cases}$$

and note that we trivially have $I(\mathcal{H})_P = \mathcal{N}_P$ a.e. for every $P \in \mathcal{L}$, hence $\phi_{\mathcal{H}}^I = \phi$.

For every $K \in \mathcal{K}_k[\text{Th}(\phi)]$, let $p_{P_K} \stackrel{\text{def}}{=} \lambda(\mathcal{H}_{P_K}) = \phi(\langle K \rangle) > 0$ and note that the definition of Θ implies that p is Θ -invariant and $\sum_{K \in \mathcal{K}_k[\text{Th}(\phi)]} p_{P_K} = 1$. To conclude the proof, we will show that $\phi_{\mathcal{H}} = \psi_{\Theta, p}$. To do so, for every $K \in \mathcal{K}_k[\text{Th}(\phi)]$, let $K_K \in \mathcal{K}_k[T_\Theta]$ be the unique model such that $\text{id}_k \in R_{P_K}(K_K)$ and note that the axioms of T_Θ imply that $W_{\mathcal{H}}^{K_K}$ is a.e. equal to the $(k-1)$ -flattening $W_{\mathcal{H}_{P_K}}^{k-1}$ of the peon $\mathcal{H}_{P_K}$, which in turn is a.e. equal to $W_{\mathcal{N}}^K$. But then from Lemma 9.4, it follows that $W_{\mathcal{H}}^{K_K} = \phi(\langle K \rangle) = p_{P_K}$ a.e. Since the T_Θ -on $\mathcal{N}^Z$ of Definition 2.9 and Proposition 9.1 also clearly satisfies $W_{\mathcal{N}^Z}^{K_K} = W_{\mathcal{N}_{P_K}^Z}^{k-1} = p_{P_K}$ a.e., from Lemma 9.3, it follows that $\phi_{\mathcal{H}} = \phi_{\mathcal{N}^Z} = \psi_{\Theta, p}$. $\square$

10 Conclusion and open problems

In this paper we have attempted to build a general theory of quasirandomness that is uniformly applicable to arbitrary combinatorial structures and is invariant under their “natural transformations”. While our basic definitions deliberately avoided mentioning specific densities, it turned out, in the vein of the previous research in the area, that our quasirandom properties can be characterized in several equivalent ways, *including* such densities. We have shown how to arrange these properties into a hierarchy and, with one or two notable exceptions, have been able to prove that this hierarchy is proper. Finally, we have compared our quasirandom properties to what has been studied before for hypergraphs (with the focus on specific densities) and have found that these two frameworks are essentially incomparable.

One topic that we touched tangentially in the proof of Theorem 3.10, more specifically with Example 1 and Lemma 7.6, is the closedness of our

properties with respect to both the density topology and L_1 -topology (Definition 7.5). The aforementioned example and lemma show that in general unique coupleability with a particular collection of limit objects is closed in L_1 -topology but not necessarily closed in the density topology. On the other hand, alternative syntactic descriptions of $\mathbf{UCouple}[\ell]$ and $\mathbf{UInduce}[\ell]$ (as ℓ -locality and symmetric ℓ -locality, respectively) imply that these classes are closed even in the density topology. So in a sense we have a satisfactory overall picture for the classes based on the “extrinsic” notion of coupleability.

Remarkably, we do not know the answer for the class $\mathbf{Independence}[\ell]$, even if it has a very clean and natural “intrinsic” definition. This is the first question we would like to ask: is $\mathbf{Independence}[\ell]$ closed in the density, or at least L_1 -topology? One sensible approach to this question might consist in developing an alternative, and perhaps more concrete, characterization of this class that might be interesting in its own right.

If ϕ_1 and ϕ_2 are uniquely coupleable with *all* theons of rank $\leq \ell$, then the same is true for $\phi_1 \otimes \phi_2$ (Theorem 3.4 (ii)). We do not know if the same remains true after replacing this class of tests with individual tests, and when we needed this in one of our proofs, we had to take a considerable detour (see item 3 in our program at the beginning of Section 7). Thus comes our second open question: assume that ϕ_1 and ψ , as well as ϕ_2 and ψ are uniquely coupleable. Does it imply that $\phi_1 \otimes \phi_2$ is also uniquely coupleable with ψ ?

Under the additional assumption that ϕ_1, ϕ_2 are themselves uniquely coupleable, the question takes a particularly nice and symmetric form: assume that ϕ_1, ϕ_2 and $\phi_3 (= \psi)$ are pairwise uniquely coupleable. Does it imply that ϕ_1, ϕ_2, ϕ_3 are (mutually) uniquely coupleable? While the analogy with independence for random variables is now visible, it is not immediately clear how useful it might turn out here.

Another interesting question is whether unique coupleability establishes a Galois correspondence between $\mathbf{UCouple}[\ell]$ and limit objects of rank at most ℓ . In other words, is it true that if ϕ is uniquely coupleable with every $\psi \in \mathbf{UCouple}[\ell]$, then $\text{rk}(\phi) \leq \ell$?

As we mentioned before, the results of Theorems 3.1, 3.2, 3.5, 3.6 and 3.7 almost complete the Hasse diagram of implications between the families $\mathbf{Independence}$, $\mathbf{UCouple}$ and $\mathbf{UInduce}$. The only missing implication/separations are the ones between $\mathbf{UCouple}[\ell]$ and $\mathbf{Independence}[\ell']$ when $\ell' < \ell$, and this is our fourth question: does $\mathbf{UCouple}[\ell]$ imply $\mathbf{Independence}[\ell - 1]$? Let us remark that with some change in geometric representation, the somewhat

subtle theons we introduced in Section 2.3 all suggest that this implication may actually hold.

Recall that Theorem 3.10(i)≡(vii) says that $\phi \in \mathbf{UCouple}[\ell]$ is equivalent to $\phi \otimes \psi_{\text{lin}} \in \mathbf{UInduce}[\ell]$. Let us now draw attention to three interesting open problems that can be extracted from this equivalence.

The first is whether a “converse” of this is true in the spirit of Theorems 3.12 and 3.13: can every $\phi \in \mathbf{UInduce}[\ell]$ be written as $\phi = (\hat{\phi} \otimes \psi_{\text{lin}})^I$ for some $\hat{\phi} \in \mathbf{UCouple}[\ell]$ and some open interpretation $I: T \rightsquigarrow T' \cup T_{\text{LinOrder}}$?

The second problem is an analogue of Theorems 3.12 and 3.13 themselves in the context of unique inducibility. We conjecture that if all arities are at most k , then $\phi \in \mathbf{UInduce}[k-1]$ should be equivalent to $\phi = (\phi_{\Theta,p} \otimes \psi_{\text{lin}})^I$ for some action $\Theta: S_k \times \mathcal{L}' \rightarrow \mathcal{L}'$ on a language $\mathcal{L}'$, some open interpretation $I: T \rightsquigarrow T_{\Theta} \cup T_{\text{LinOrder}}$ and some Θ -invariant $p \in [0, 1]^{\mathcal{L}'}$ (of course, this would follow from a positive answer to the previous problem).

The third question is more open-ended. In the three scenarios discussed in Section 3.1 (permutations, words and Latin squares), the quasirandom object is “straightforward” but does not satisfy even the weakest of our properties $\mathbf{UInduce}[1]$. Hence we might reasonably ask if the theory of “natural” (understood as in the introduction) quasirandomness properties can be extended beyond $\mathbf{UInduce}[1]$. One possibility would be to consider the closure of $\mathbf{UInduce}[\ell]$ under independent couplings and open interpretations. Both the quasirandom permuton $\psi_{\text{lin}} \otimes \psi_{\text{lin}}$ and the quasirandom Latin square $\psi_{\text{lin}} \otimes \psi_{\text{lin}} \otimes \psi_{\text{lin}}$ belong to this class (for every ℓ). This definition, however, is of the same distinctly ad hoc nature we have been trying to avoid in this paper. Are there any “reasonable” descriptions of this class, be them extrinsic or intrinsic? The only thing we can prove (and even that is non-trivial) is that this class is proper, i.e., there are theons that do not belong to it, for an arbitrary ℓ . If the conjectures from the previous two paragraphs are true, this would also form another interesting hierarchy: starting from $\mathbf{UCouple}[\ell]$, we can get progressively weaker families of natural quasirandomness properties by taking independent coupling with the linear order ψ_{lin} .

Another possible approach would be to start with quasirandom permutations that is by far the most widely studied class, and from their known properties [Coo04, Coo05, KP13, CKN⁺20]. However, in comparison to their (hyper)graph and tournament counterparts, the theory of permutation quasirandomness provides a much smaller variety of quasirandomness formulations as candidates for natural generalizations, essentially boiling down to only

three types: explicit density notions, discrepancy notions based on intervals and spectral notions. Let us also note that there is still a whole host of properties [DEG14, CD17] that random permutations satisfy and that have not yet been fully explored in the quasirandom setting. In fact, some of these properties are so fine-grained that it is not even clear if they can be encoded by subpermutation densities.

The notions of rank and **Independence** have the following generalization: for $B \subseteq \mathbb{N}_+$, let us say that a peon $\mathcal{N}$ over $\Omega = (X, \mathcal{A}, \mu)$ is *B-compatible* if it only depends on coordinates that are indexed by sets A with $|A| \in B$, that is, it can be written as $\mathcal{N} = \mathcal{G} \times X^{\bigcup_{b \in [k(P)] \setminus B} \binom{[k(P)]}{b}}$ for some $\mathcal{G} \subseteq X^{\bigcup_{b \in B} \binom{[k(P)]}{b}}$. Let us say that an Euclidean structure is *B-compatible* if all its peons are so and let us say that $\phi \in \text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ is *B-compatible* if it has a T -on representation that is *B-compatible*. Then rank at most k amounts to $[k]$ -compatibility and ℓ -independence amounts to $(\mathbb{N}_+ \setminus [\ell])$ -compatibility. We believe that with a careful inductive application of the theon uniqueness theorems [CR20, Theorems 3.9 and 3.11, Proposition 7.7], one could generalize the proof of weak independence to show that if ϕ_1 and ϕ_2 are B_1 -compatible and B_2 -compatible, respectively and $B_1 \cap B_2 = \emptyset$, then ϕ_1 and ϕ_2 are uniquely coupleable. However, we know that $\text{UCouple}[\ell]$, i.e., unique coupleability with all $[\ell]$ -compatible limit objects, is strictly weaker than **Independence** $[\ell]$, so it is natural to ask if the weak independence analogue of $(\mathbb{N}_+ \setminus B)$ -compatibility (i.e., asking the exchangeable array $\mathbf{K}$ to be independent from $(\theta_A \mid |A| \in B)$ as a random variable) also yields a strictly weaker property than $(\mathbb{N}_+ \setminus B)$ -compatibility when B is not of the form $[k]$ for some $k \in \mathbb{N}$. In particular, this involves studying unique coupleability with all $\psi \in \text{Independence}[\ell]$ as well. Building on that, it is also natural to ask if there are examples of uniquely coupleable ϕ_1 and ϕ_2 that do not fall in this *B*-compatibility setting or in its weak independence analogue.

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A The L_1 -topology

Lemma A.1. *The L_1 -distance δ_1 is a metric on $\text{Hom}^+(\mathcal{A}[T], \mathbb{R})$ and generates a finer topology than the density topology.*

Proof. Let us first check the triangle inequality. Let ξ be a coupling of ϕ_1 and ϕ_2 and ζ be a coupling of ϕ_2 and ϕ_3 attaining the L_1 -distances in (30). Let also $J_i: T \rightsquigarrow T \cup T$ be the structure-erasing interpretation corresponding to coordinate i and $I_{ij}: T \cup T \rightsquigarrow T \cup T \cup T$ be the structure-erasing interpretation corresponding to coordinates i and j . Since ξ is a coupling of ϕ_1 and $\phi_2 = \zeta^{J_1}$, Proposition 5.2 gives us a coupling $\hat{\xi}$ of ϕ_1 and ζ such that $\hat{\xi}^{\text{id}_T \cup J_1} = \xi$. Since $\text{id}_T \cup J_1 = I_{12}$, we get that $\hat{\xi}$ is a coupling of ϕ_1 , ϕ_2 and ϕ_3 such that $\hat{\xi}^{I_{12}} = \xi$ and $\hat{\xi}^{I_{23}} = \zeta$. But $\hat{\xi}^{I_{13}}$ is a coupling of ϕ_1 and ϕ_3 and for each $P \in \mathcal{L}$ we have

$$\hat{\xi}^{I_{13}}(d_P) \leq \hat{\xi}^{I_{12}}(d_P) + \hat{\xi}^{I_{23}}(d_P),$$

hence by (30) we get $\delta_1(\phi_1, \phi_3) \leq \delta_1(\phi_1, \phi_2) + \delta_1(\phi_2, \phi_3)$.

Finally, note that by (29) we have

$$|\phi_1(\langle M \rangle) - \phi_2(\langle M \rangle)| \leq \delta_1(\phi_1, \phi_2) \sum_{P \in \mathcal{L}} (|M|)_{k(P)},$$

for every $M \in \mathcal{M}[T]$. This implies both $\delta_1(\phi_1, \phi_2) = 0 \implies \phi_1 = \phi_2$ and that the L_1 -topology is finer than the density topology. $\square$