# The Sources of Certainty in Computation and Formal Systems

Michael J. O'Donnell
*The University of Chicago*
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computation = derivation in formal system

Familiarity makes a difference

- computational is a synonym for formal, not a second topic

- familiarity with computation makes ideas, formerly arcane and subtle, more accessible

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What is a *formal* system?
Consider its opposite.

formal vs.

~~intuitive~~

~~casual~~

~~relaxed~~

~~unrigorous~~

~~incomplete~~

contentual

- *formal*: concerned with form, opposite of
- *contentual*: concerned with content
- *contentual* is not common English, translation of German *inhaltlich*

$\boxed{\text{0 min}}$

Practice of

Formalism

(use of formal systems)

| | | |
|---|---|---|
| Form | | Content |
| | versus | |
| ceremony | | communication |

## Computer science and other disciplines

$$\text{CS} \xrightarrow{\text{calculate}} \text{Physics}$$

$$\text{CS} \xleftarrow{\text{survey}} \text{Soc Sci}$$

$$\text{CS} \xrightarrow{\text{explain}} \text{Philosophy}$$

computations about —

vs.

computations in the *content* of —

- **Common:** CS serves other disciplines by performing computation

- **Common:** Other disciplines study systems containing electronic computers

- **New:** CS serves other disciplines by describing the computations that occur unconsciouly in them
  - genetic code as programming language
  - immunology as control system, digital signature, pattern-matching
  - information theory applied to thermodynamics
  - potential computational characterization of limits of quantum coherence

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– complexity theory in thermodynamics, probability

*What the Tortoise Said to Achilles*
Lewis Carroll
*(red stuff is mine)*

. . .

> (D) If [A and B] and C are true, then Z must be true.

. . .

Achilles triumphantly replied: "Logic would tell you 'You can't help yourself. Now that you've accepted A and B and C and D, you must accept Z!' So you've no choice, you see."
"Whatever Logic is good enough to tell me is worth writing down," said the Tortoise. "So enter it in your book, please. We will call it

> (E) If [A and B and C] and D are true, Z must be true."

Gentzen system

*3 levels of implication*

$$\frac{\Gamma, \alpha \Rightarrow \beta \vdash \alpha, \Psi \qquad \Gamma, \beta, \alpha \Rightarrow \beta \vdash \Psi}{\Gamma, \alpha \Rightarrow \beta \vdash \Psi}$$

*(or 4 including the discussion of the system)*

Brouwer-Heyting-Kolmogorov Interpretation

*(quoted from Wikipedia, red stuff mine)*

A proof of $P \Rightarrow Q$ is a *definition of a* function $f$ which converts a proof of $P$ into a proof of $Q$.

Including a proof that $f$ is well-defined and performs the conversion,

**and . . .**

Commment from *Platonic Realms Interactive Mathematics Encyclopedia*

. . . modern mathematics relies ultimately on pure formalism in its use of logic. This avoids the infinite regress in which the Tortoise traps Achilles.

This trap is impossible to avoid if logic is not formalized, because . . . in order to know how to use a rule (such as a rule of inference) you need a rule telling you how to apply the rule. . . . By contrast, in formal logic, rules of inference are reduced to rules of symbol manipulation. Since the symbols themselves are uninterpreted (which is what we really mean by "formal"), we have a system as austere and elegant as chess, where it is understood that the game arises from—and entirely consists in—the rules for moving the pieces on the board.

I find every detail of this passage seriously wrong, including the understanding of chess.

1. The infinite regress applies equally to formal and contentual interpretations.

2. One may (in fact, must due to finite lifetime) break the regress either formally or contentually.

3. Carroll shows, not that either formal or contentual reasoning fails, but that neither has an unquestionable foundation.

$\boxed{\text{2 min}}$

*Gödel, Escher, Bach*, p. 170
Douglas Hofstadter

. . . the trap was the idea that before you can use any rule, you have to have rule which tells you how to use that rule; in other words, there is an infinite hierarchy of levels of rules, which prevents any rule from ever getting used . . . However, we all know that these paradoxes are invalid, for rules do get used . . . How come?

Hofstadter here objects to a "Jukebox" theory of meaning, comparing it to the structure of the Carroll paradox. I don't think he intended to capture Carroll's point here.

*Gödel, Escher, Bach*, pp. 192-193

. . . You can't go on defending your patterns of reasoning forever. . . .

A system of reasoning can be compared to an egg. An egg has a shell which protects its insides. If you want to ship an egg somewhere, though, you don't rely on the shell. You pack the egg in some sort of container . . . . However, no matter how many layers . . . , you can imagine some cataclysm which could break the egg. But that doesn't mean that you'll never risk transporting your egg.

The Tortoise doesn't show that you can't apply rules. Rather, he shows that there is an infinite regress of justifying rules.

$\boxed{\text{2 min}}$

## *Zeno* vs. *Carroll*

Both paradoxes generate an infinite *d*iscourse.

- **Zeno:** infinite description of finite event.
- **Carroll:** infinite attempt to justify finite event.

In both cases, the event can happen, but the description/justification fails. That matters more for the justification than for the description.

## A formal system for incrementing integers

$$\Longrightarrow \quad x = x$$

$$0\uparrow \quad \Longrightarrow \quad 1$$

$$1\uparrow \quad \Longrightarrow \quad \uparrow 0$$

$$=\uparrow \quad \Longrightarrow \quad =1$$

- sequences of 0, 1, =, $\uparrow$
- $\Longrightarrow$ is a metasymbol (descriptive)
- $x$ stands for any sequence of 0, 1, $\uparrow$
- formality resides in content of this description, not its form
- English description doesn't diminish formality of the system described
- formal system for describing formal systems also exists

$\boxed{\text{2 min}}$

## Formal derivation of $3 + 1 = 4$
## ($11 + 1 = 100$ in binary)

$$11\!\uparrow = 1\,\boxed{1\!\uparrow}$$

$$11\!\uparrow = \boxed{1\!\uparrow}\,0$$

$$11\!\uparrow \boxed{= \uparrow}\,00$$

$$11\!\uparrow = 100$$

- all grade school arithmetic can be done this way

- 2-dimensional rules are equally feasible

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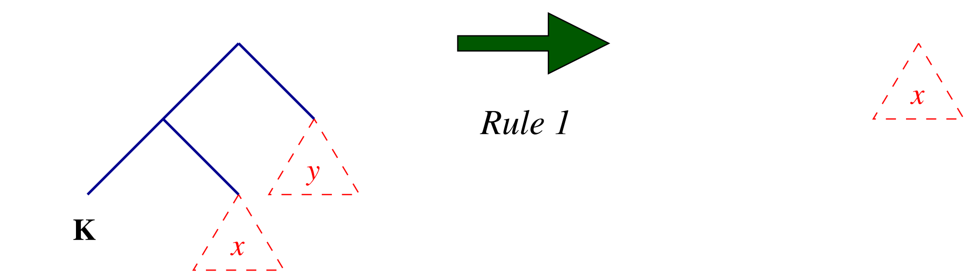
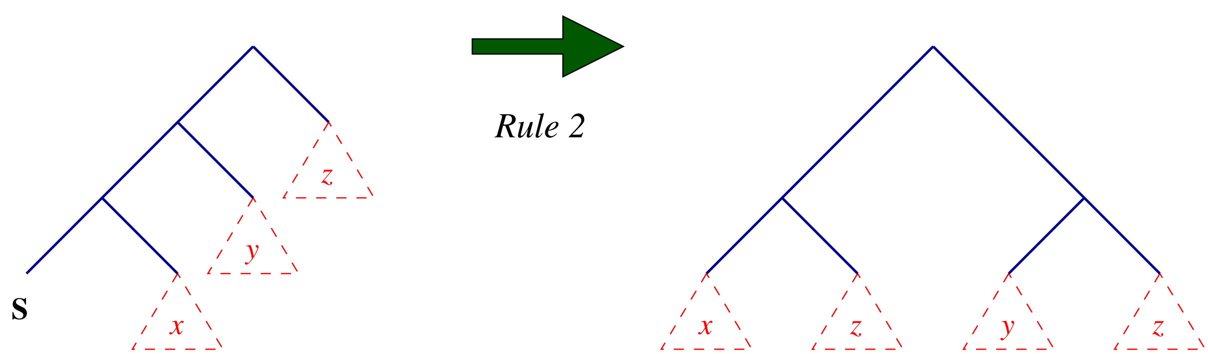
# Formal rules for the *Combinator Calculus*



- binary branching tree graphs with $\mathbf{S}$, $\mathbf{K}$ at "leaves"

- dashed triangles with $x$, $y$, $z$ are *descriptive* (meta) variables, not part of system

- replace any substructure according to the rules

- $\mathbf{K}$ is konstant operator

- $\mathbf{S}$ does a weird shuffle

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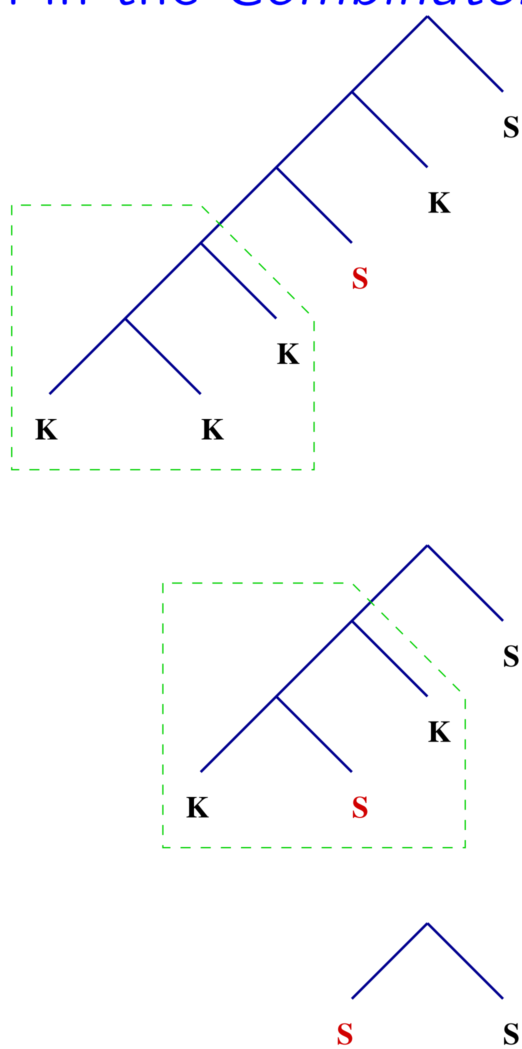
# Derivation in the *Combinator Calculus*



- dashed boxes show where left side of rule occurs

- leftmost two $\mathbf{K}$s select the red $\mathbf{S}$

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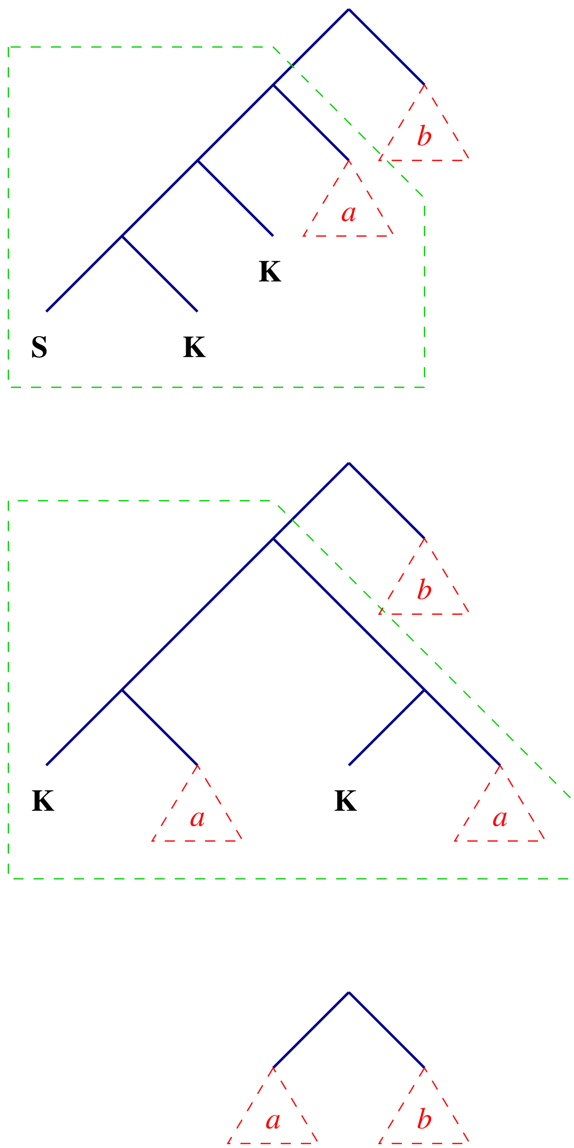
## Identity function in the *Combinator Calculus*



- leftmost **SKK** act as identity operator applied to $x$

- $x$ and $y$ are schematic/pattern variables

- this figure is not a derivation—it stands for an infinite class of derivations

- Combinator Calculus contains the behavior of every formal system

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What is the content of a formal system?

~~ink on paper~~

vs.

abstract structure, representable by

ink on paper

chalk on slate

electrons on phosphors

vibrations in air

neural signals

- the content of a formal system is abstract formal structure, not a specific physical presentation (as often misconceived)

- **evidence:** we easily switch medium

- **significance:** we can choose the most effective medium

- **nonetheless:** a formal system is real and objective, but not physical

- mental construct is a uniquely crucial presentation, but still just another presentation

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## How do formal systems occur in mathematics?

- ~~mathematics is a formal game~~
  - "The Unreasonable Effectiveness of Mathematics"
    *E. P. Wigner, R. W. Hamming*
    (according to vulgar formalists)

vs.

- mathematics studies qualities of formal systems
  - "Mathematics is the science of formal systems."
    *H. B. Curry*
  - "The content of mathematics is form."
    *me*
  - "Functional formalism"
    *S. Mac Lane*

- “vulgar formalism”: mathematics is (misconceived) a formal game

- Wigner & Hamming do not espouse vulgar formalism

- Curry’s & my slogans, vs. Mac Lane’s thorough analysis

- **important:** formal systems are real, objective, not physical, accessible to observation/study, crucially involved in all mathematical content

- **not needed:** precise characterization of *mathematics*

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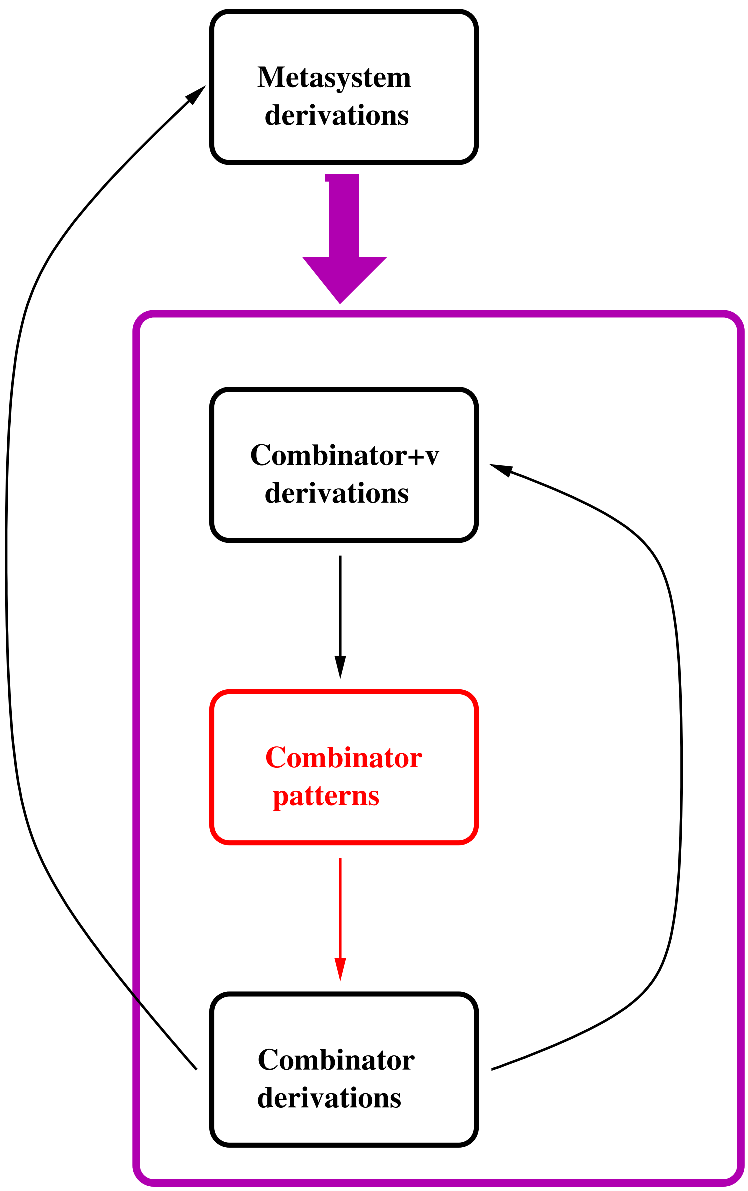
## Reflexive modeling relations (highly simplified)



- start with derivations in Combinator Calculus

- observe schematic patterns in CC derivations

- new formal system—CC+variables—contains behavior of CC patterns in its individual derivations

- CC derivations model behavior of CC+variables

- another metasystem models all of the above & their relations

- CC models the metasystem

- reflexivity is very powerful & naturally confusing

$\boxed{\text{10 min}}$

- vulgar formalism arises from reflexive confusion, but requires infinite regress of modeling relations ("What the tortoise said to Achilles," by Lewis Carroll)

- universality of CC provides a single language for modeling, but doesn't avoid infinite regress of modeling relations

What if formal systems were designed by a *conceptual engineer*

- Minimize physical cost of manipulations
  - Use symbols—choose presentation to avoid manipulation cost
- Maximize objective certainty in the conclusions
  - Use formal rules—choose presentation to avoid ambiguity

- the concept of formal systems has the quality of an engineering product, even though derived by social evolution

- even the result of the engineering is a process, more than a static object

- **negative quality:** we achieve certainty by refusing to use symbols and relations unless all parties agree on their significance

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## Descartes and Hilbert

"I was given to believe that . . . a clear and certain knowledge of all that is useful in life might be acquired."

—*René Descartes* (1637)

"I should like to eliminate once and for all the questions regarding the foundations of mathematics . . . , thus recasting mathematical definitions and inferences in such a way that they are unshakable and yet provide an adequate picture of the whole science."

—*David Hilbert* (1927)

- two calls for certainty, in everything, then in mathematics alone

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## Descartes' method—task specification for design of formal systems?

The method:

1. "never to accept anything for true which I did not clearly know to be such"

2. "to divide each of the difficulties under examination into as many parts . . . as might be necessary for its adequate solution"

3. "to conduct my thoughts in such order [from] the simplest and easiest to know, . . . to the knowledge of the more complex"

4. "to make enumerations so complete . . . that I might be assured that nothing was omitted"

- nothing in the language explicitly calls for formality

- it's all consistent with an engineering task specification leading to formal systems

- there are no competing bids

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## Descartes' method includes formal geometry

"I observed, that the great certitude which by common consent is accorded to [geometric] demonstrations, is founded solely upon this, that they are clearly conceived in accordance with the rules I have already laid down."

- geometry is a formal system, but Descartes may not have recognized it as such

- Descartes fingers geometry as the only clear example of successful application of his method

- this is good evidence that Descartes' program for certainty is formal, at least where it applies to mathematical knowledge

$\boxed{\text{13 min}}$

What about “cogito”?

“I think, therefore I am.”

An unsuccessful attempt to introduce a new postulate with the same certainty *mistakenly* attributed to the postulates of geometry.

- Descartes and contemporaries recognized certainty in geometric derivations

- they also attributed certainty to geometric postulates—this was an error

- Descartes is trying—I think failing—to develop another postulate with the certainty attributed falsely to the geometric postulates

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## Hilbert's program—explicitly formal

"In my theory contentual inference is replaced by manipulation of signs according to rules; in this way the axiomatic method attains . . . reliability and perfection."

"A formalized proof, like a numeral, is a concrete and surveyable object."

"We recognize that we can obtain and prove [numerical] truths through contentual intuitive considerations."

- Hilbert is quite explicit that formal systems uniquely satisfy his requirement for certainty

- formal derivations ("proofs") are concrete objects to be observed

- Hilbert emphasizes the contentual appreciation of numbers, but numbers themselves are forms

- Hilbert's emphasis on the content of numbers is probably a defense against other current views

- Hilbert recognized the concept of the *infinite* as one requiring new formal support

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## Asessing formal certainty

- How certain?
  - $\geq$ the sky is blue
- What flavor of certainty?
  - ~~Gibraltar will stand~~
  - we can drive to the store
- Certain of what?
  - ~~measurements are real numbers~~
  - correctness of derivations, patterns
- How robust is our certainty?
  - as good as it gets

- certainty about formal derivations is arguably the strongest certainty achievable

- formal systems contribute nothing directly to certainty about the nonformal content of postulates

- the certainty can cover essentially all observations about formal patterns

- directly observed patterns often formally entail other patterns that we don't notice directly

- reflexivity magnifies the power of formalism

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## Formal limits on formal certainty

- Gödel: no single formal system for integer number theory
  - Hilbert's program impossible
  - Descartes' larger program impossible

- Gödel: no formal system powerful enough to analyze its own correctness
  - infinite regress? maybe not
  - Takeuti: ontological level $\neq$ power

- the reflexivity of formal studies leads to formal limits on the power of formalism

- Gödel's 1st incompleteness: Hilbert's & Descartes' programs impossible

- 1st incompleteness limits the scope of certainty

- Gödel's 2nd incompleteness is not necessarily a limit on certainty—ontologically primitive systems might have great formal power (Gaisi Takeuti)

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## Summary

Formal systems are real, objective, not physical

- certainty in process, not static
- certainty from choice of presentation
- about patterns/relations, not physical facts
- reflexivity expands the power of formality
- insight into Descartes' Hilbert's programs
- form is very important sort of content