# The Sources of Certainty in Computation and Formal Systems

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2 November, 1999
Bibliography corrected 28 February, 2001

**Abstract**

In his *Discourse on the Method of Rightly Conducting the Reason, and Seeking Truth in the Sciences*, René Descartes sought "clear and certain knowledge of all that is useful in life." Almost three centuries later, in "The foundations of mathematics," David Hilbert tried to "recast mathematical definitions and inferences in such a way that they are unshakable." Hilbert's program relied explicitly on *formal systems* (equivalently, *computational* systems) to provide certainty in mathematics. The concepts of computation and formal system were not defined in his time, but Descartes' method may be understood as seeking certainty in essentially the same way.

In this article, I explain formal systems as concrete artifacts, and investigate the way in which they provide a high level of certainty—arguably the highest level achievable by rational discourse. The rich understanding of formal systems achieved by mathematical logic and computer science in this century illuminates the nature of programs, such as Descartes' and Hilbert's, that seek certainty through rigorous analysis.

I presented this paper on 30 October 1999, at the 1999–2000 Sawyer Seminar at the University of Chicago, *Computer Science as a Human Science: the Cultural Impace of Computerization.*

# 1 Scholarly Disclaimer

I use the writings of several excellent thinkers—Descartes, Hilbert, Curry, Mac Lane—to stimulate insight regarding the deliberate and accidental use of formal systems in philosophical and mathematical thought. I believe that all of my quotes are interpreted sensibly, but I am *not* trying to infer the actual beliefs or intentions of the authors. Nor am I trying to make a fair representation of the whole of their works. Rather, I am taking the writings of these authors as tools to be applied at need, and selecting only those writings that relate to my topic.

The modern authors in my list are all well versed in the concept of formal system, and it is clear that they intended to discuss such systems, whether or not they would agree with my own use of those discussions. Descartes had no access to a definition of formal system (although he focused much attention on Euclidean geometry, which we recognize now as a formal system). I will argue that it makes sense, in retrospect, to apply many of Descartes' ideas to formal systems, but I make no claim that Descartes himself would have done so under some sort of counterfactual assumptions.

My topic, then, is the illumination of certain lines of thought using the concept of formal systems. I welcome the contributions of great thinkers to that topic, whether those contributions are intentional or accidental.

# 2 Computational Concepts in Uncomputational Topics

This article constitutes part of a larger program of my own, to connect computer science to other disciplines in an unusual way. I am familiar with interdisciplinary work

- that uses computations on electronic devices to calculate the consequences of various theories, or
- that applies the methods of other disciplines to study the behavior of systems comprising computers and other entities (such as people).

I intend a third form of connection between computer science and other disciplines

- that applies *concepts* from computer science to illuminate phenomena studied by other disciplines.

Computation is a sort of behavior, and the electronical gizmos that we now call "computers" are not the only devices that exhibit that sort of behavior. A few decades ago, "computer" was a job title for a person, and Alan Turing called his proposed gizmo an "Automated Computing Engine," to distinguish it from a human "computer." Other systems in the world may exhibit computational behavior unconsciously. The concept of computation was developed quite clearly before the construction of automatic computers. But, familiarity with the behavior of automatic computers makes some ideas, that once seemed highly arcane and subtle, intuitively accessible to a much wider range of thinkers than before.

Computational concepts may now be applied to understand phenomena in quantum physics, thermodynamics, probability, genetics, and immunology. In this article, I follow another such application to philosophy.

# 3 Formal Systems and Computation

The phrase *"formal system"* is widely misunderstood, even by mathematicians who profess formalist foundations for their work. Perhaps a quick way to refine the understanding of "formal" is to consider its opposite. In common use, the opposite of "formal" might be "casual," or "relaxed," or "unrigorous." Mathematicians often call a derivation "informal" if it is incomplete or not described quite thoroughly. None of these is a sensible opposite to "formal" for our purposes. A formal system is one that deals with the forms of arrays of symbols and relations between them—a proper opposite is a *contentual* system that deals with the content or meaning of arrays of symbols.

Here is an example description of a formal system capturing a tiny bit of mathematics.

**Example 1 (A formal system for incrementing integers.)**
Figure 1 describes derivation rules for a simple formal system. The left column presents the rules schematically, and the right column provides an alternate presentation in English. In the formal system described above, we

| | | | |
|---|---|---|---|
| | $\Longrightarrow$ | $x = x$ | You may start with two copies of the same sequence of '0's '1's and '↑'s, with '=' between them. |
| $0\uparrow$ | $\Longrightarrow$ | $1$ | You may replace the pair '0' followed immediately by '↑' with '1'. |
| $1\uparrow$ | $\Longrightarrow$ | $\uparrow 0$ | You may replace the pair '1' followed immediately by '↑' with '↑0'. |
| $=\uparrow$ | $\Longrightarrow$ | $=1$ | You may replace '↑' by '1' when it follows immediately after '='. |

Figure 1: Derivation rules for a formal system for incrementing integers

may derive '$11\uparrow = 100$' as follows:

$$\begin{aligned}11\uparrow &= 11\uparrow\\ 11\uparrow &= 1\uparrow 0\\ 11\uparrow &= \uparrow 00\\ 11\uparrow &= 100\end{aligned}$$

**End Example 1** □

The formal system of Example 1 may be understood as a presentation of the theory of nonnegative integers expressed in binary notation, with the operation of adding one indicated by placing '↑' to the right of an integer. The derivation in the example may be understood to demonstrate, in more familiar notation, that $3 + 1 = 4$.

One more example suggests the variability and power of formal systems, at the cost of appearing very esoteric in interpretation.

**Example 2 (A simple formal system capable of all computation.)** Figure 2 shows the two rules for manipulation in a famous formal system called the *Combinator Calculus*.

The English description of the rules is too long and tangled to be worth inspecting here. The pictures should be clear enough, as long as we understand that

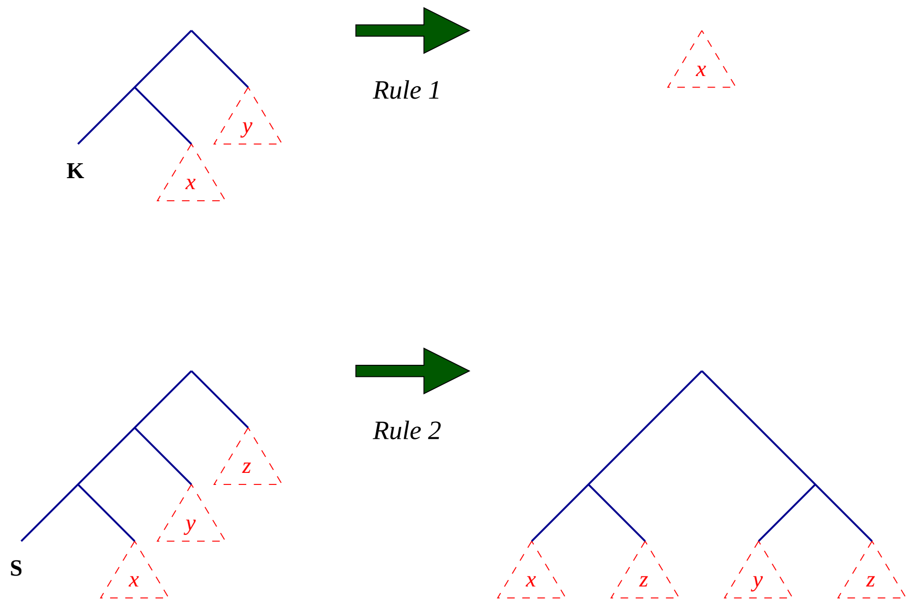
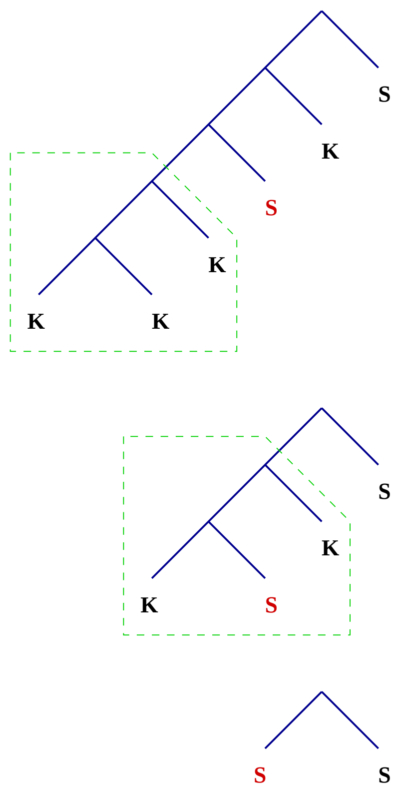



Figure 2: Derivation rules for the Combinator Calculus

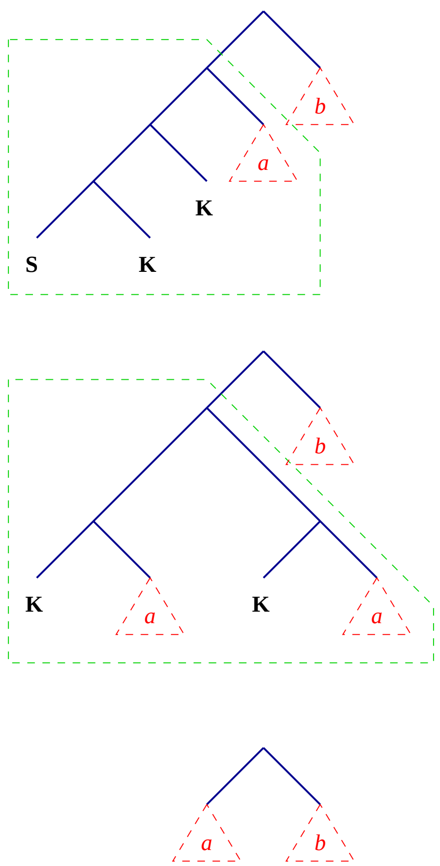
- The system deals entirely with finite binary branching tree diagrams, where the end of each path is labelled with exactly one of the symbols '**S**' or '**K**'. Such a tree diagram is called a *combinator*.
- You may start with any combinator.
- In Figure 2, the $x$, $y$, and $z$ in dashed triangles may be replaced by any combinators, as long as in each application of a rule, each of the $x$ triangles is replaced by a copy of the same combinator, similarly for each of the $y$ triangles and each of the $z$ triangles.
- When a structure of the form given by the left-hand side of one of the two rules in Figure 2 appears anywhere within a combinator, you may replace that structure by the corresponding structure on the right-hand side of the same rule.

In the formal system of the Combinator Calculus, we may replace a certain combination of four '**K**'s and two '**S**'s by the combination of the two '**S**'s, using the derivation in Figure 3. We may always eliminate a certain combination of an '**S**' and two '**K**'s, as shown in the schematic derivation of Figure 4.




This derivation takes two steps with *Rule 1* to derive a tree with two **S**s. Intuitively, the leftmost two **K**s select the red **S** from the surrounding **K**s. The five-sided dashed figures are not part of the derivation: they just show where the rules are applied.

Figure 3: A derivation in the Combinator Calculus




This schematic derivation takes two steps, first with *Rule 2* and then with *Rule 1*, to reach the schematic tree at the bottom. Intuitively, the combination of one **S** and two **K**s above acts as an *identity operator* applied to the tree that fills in for $a$.
The five-sided dashed figures are not part of the derivation: they just show where the rules are applied.

Figure 4: A schematic derivation in the Combinator Calculus

**End Example 2** □

The useful interpretations of the Combinator Calculus are too long to describe here. The interesting qualities of this formal system for our purposes are

- it is best described in terms of binary branching tree diagrams, instead of sequences of symbols;
- although it has only two rather simple rules, it is capable of deriving values of every function that is computable by every other formal system.

The derivation in Figure 3 shows how two '**K**'s may be used to select the middle of a sequence of three structures (in this case, the middle structure is just an '**S**', marked red to help you follow the process). Notice that the regions in the green dashed lines contain the pattern in the left-hand side of *Rule 1*.

1. In the first step, the leftmost **K** matches the **K** in the left-hand side of *Rule 1*, and the next two **K**s to the right fill in for the $x$ and $y$. Following *Rule 1*, this whole structure with three letters is replaced by **K** (filling in for $x$ in the rule).
2. In the second step, the leftmost **K** matches the **K** in the left-hand side of *Rule 1*, the red **S** fills in for the $x$, and the second **K** fills in for the $y$. Following *Rule 1*, this whole structure with three letters is replaced by **S** (filling in for $x$ in the rule).

We quickly get bored doing one derivation at a time, and notice *schematic derivations*—patterns that can be expanded into infinitely many different but similar derivations. The schematic derivation in Figure 4 shows how an appropriate combination of an '**S**' and two '**K**'s represents the identity function: when applied to any combinator $a$ it produces (after two derivation steps) $a$ alone. Figure 4 is *schematic* in the sense that, although it does not present a derivation per se, every systematic replacement of the triangles containing $a$ and $b$ in the figure yields a derivation.

1. In the first step, the leftmost **S** matches the **S** in the left-hand side of *Rule 2*, the two **K**s fill in for $x$ and $y$, and whatever fills in for $a$ also fills in for $z$ in the rule. Following *Rule 2*, this whole structure is replaced by the combination of two **K**s and two $a$s.

2. In the second step, the leftmost **K** matches the **K** in the left-hand side of *Rule 2*, the leftmost $a$ fills in for $x$, and the second combination of **K** with $a$ fills in for $y$. This whole structure is replaced by a single copy of $a$.

The idea of a schematic derivation is worth some attention, as it illustrates the highly reflexive way in which formal systems provide reasoning power. Most of the intuitively important observations about formal systems are schematic—they are observations of *patterns* in the derivations of the formal system, rather than individual derivations. But, there is another formal system containing the derivations of the Combinator Calculus, and also derivations with formal variable symbols. Individual derivations in the Combinator Calculus with variables correspond to schematic patterns of derivations in the Combinator Calculus, in a rigorous way. There is yet another formal system that models the correspondence between schematic derivations in the Combinator Calculus and derivations in the Combinator Calculus with variables.

But, the trickiest twists are yet to come. The Combinator Calculus was designed specifically to be able to simulate the behavior of systems with variables, in a variable-free style. So, the Combinator Calculus contains a precise model of the behavior of the Combinator Calculus with variables, and therefore single derivations in the Combinator Calculus can demonstrate the behaviors of schematic derivations in the Combinator Calculus. And, there's a formal system that models the correspondence between the Combinator Calculi with and without variables, and the Combinator Calculus contains a model of that system, and .... Figure 5 suggests the systems and relations described above, but of course the real picture is infinitely large, and infinitely more complicated.

Formal systems can be used to study one another in highly tangled and reflexive, but also powerful and productive, ways. This tangled reflexivity has a lot to do with the effectiveness of formal methods, and it is no doubt the source of a lot of the confusion about the precise relationship between formal systems, mathematics, and the rest of the world. Saunders Mac Lane traces the reflexivity of formal systems within mathematics rather thoroughly in *Mathematics, form and function.*

Those who pay attention to such things appear to accept unanimously that

- formal systems, such as the one described in Example 1, may be treated

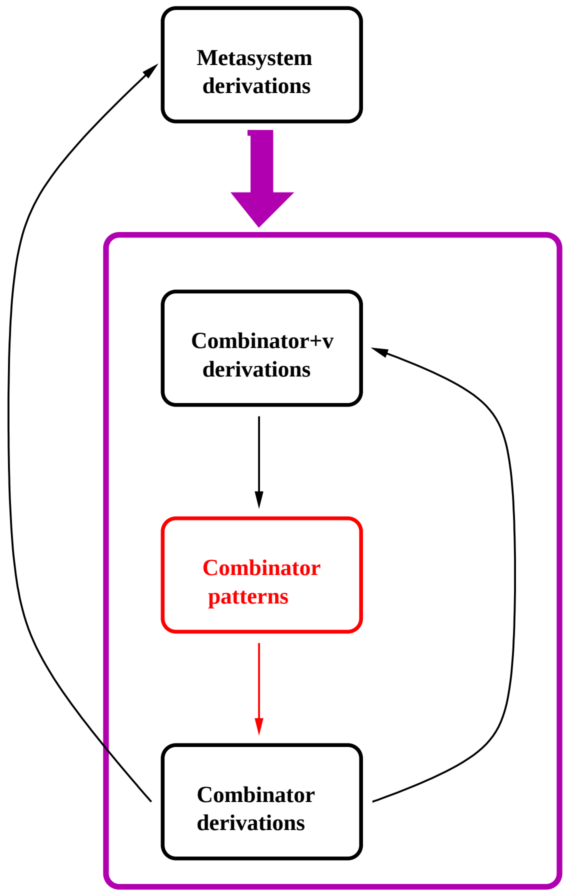



Figure 5: Variations on Combinator Calculus and their modeling relations

as pure symbolic games, without referring to any meaning that might be associated with their symbols; and

- formal systems are at the heart of mathematics.

Unfortunately, many thinkers (including working mathematicians) also appear to misconstrue the detailed nature of formal systems, and their particular relation to mathematics.

**Symbols and form as physical objects, vs. abstract constructs.** A natural and common view of formal systems holds that they describe physical operations that may be applied to ink on paper, or at least to specifically typographical presentations of symbols with specific geometrical shapes. This view makes formal systems satisfyingly concrete and physical at first glance, but it also makes them very puzzling, since the particular qualities of, for example, ink spread on paper, have little to do with the interesting qualities of formal systems. The actual practice of presenting formal systems argues against a specific physical view. Practitioners of formal studies routinely accept and appear to understand presentations with ink on paper, chalk on slate, electron beams bombarding phosphors, sound vibrations in the air, and electrochemical systems in the brain. The mental presentation is uniquely important, since it alone is irreplacable when we use formal systems as thinking tools. But, in essence, it is still just another presentation.

Formal systems surely have to do with the forms of arrays of symbols, rather than the meanings of such arrays. But, the symbols and arrays are best understood as mental constructs, which may be communicated through any physical presentation that satisfies all the parties to a discussion. Haskell Curry explains this view rather well in *Outline of a Formalist Philosophy of Mathematics*. Curry even challenges the common notion that arrays of symbols should always be presented as linear sequences of characters, a widely accepted view that appears to me to derive accidentally from typographical technology. He argues that graphical structures, usually in the form of trees such as those in Example 2, are more suitable for most applications of formal systems in mathematics. In principle, formal systems should be defined using whatever sorts of layouts of symbols are most effective for their particular uses.

**Mathematics as a formal system, vs. mathematics as a study of formal systems.** The second misconstrual of formal systems has to do with their particular relation to mathematics. Many thinkers, especially working mathematicians, appear to accept the view that mathematics is a game played out by the rules of some formal system, or perhaps a collection of such games. That is, they accept the notion that mathematicians at work are carrying out derivations in a formal system. A variation on this view holds that, while mathematicians at work may employ mental steps that do not follow the rules of any well identified formal system, the correctness of that work depends on the possibility of recasting it as such a formal derivation. The actual practice of mathematics is viewed as a slightly risky, but more efficient, prediction of the results of particular formal derivations. Mac Lane calls the notion of mathematics as an arbitrary formal game "vulgar formalism." I cannot find an explicit profession of "vulgar formalism," even with a more dignified title, in print. But, I have heard mathematicians and users of mathematics express such views in casual conversation, and they sounded serious.

If mathematics is indeed the playing out of a game whose rules have only to do with the superficial forms of symbols, and nothing to do with their useful meanings, we are naturally puzzled why mathematics appears to be useful as a tool for practical work. R. W. Hamming and E. P. Wigner expressed this puzzlement in papers called "The unreasonable effectiveness of mathematics" and "The unreasonable effectiveness of mathematics in the natural sciences," respectively, whose titles are quoted widely by mathematicians who worry that a mere formal system should not capture scientific ideas with real-world content. I think that it is much more sensible to view mathematics as a study with real objective content, whose effectiveness is a result of the widespread applicability of that content. (Hamming and Wigner do not seem to hold a formalist view of mathematics themselves, and their papers contain a lot of interesting discussion of the actual practice of mathematics and science.)

Rather than mathematics itself being a formal game, I support the view that mathematics is essentially a rigorous study of the behavior of formal systems. Confusion about the relation of mathematics to formal systems probably derives largely from the way in which mathematical tools are particularly effective for studying technical methodological issues in the practice of mathematics itself. For example, in mathematical logic we study formal systems that are designed to illuminate mathematical reasoning. It is not hard to misconstrue this application of formal tools to the study of mathematics as an a priori embedding of mathematics into a formal system.

Saunders Mac Lane explains the role of formal systems in mathematics very thoroughly in *Mathematics, Form and Function*, including the convolutions induced by the reflexive uses of mathematics. Haskell Curry covered similar ground earlier in *Outline of a Formalist Philosophy of Mathematics*. I find Curry's discussion less satisfying in its treatment of actual mathematical activity, but he contributes a slogan which I find extremely useful as long as it is used to stimulate thought, rather than as a final conclusion:

> Mathematics is the science of formal systems.

That is, mathematics is not just the playing of a formally defined game; it has objective content—the qualities of formal systems. I propose another slogan to stimulate similar thought from a slightly different point of view:

> The content of mathematics is form.

For present purposes, it is important that formal systems are real, objective, but not physical, things, that we can be highly certain about their behavior, and that the objective study of formal systems is at least a large part of the content of mathematics. But, it is not important to maintain Curry's slogan, or mine, as a precise characterization of everything in the practice of mathematics. Read *Mathematics, Form and Function* for a careful discussion of the rich variety of activities involved in the actual practice of mathematics.

The reflexivity of formal systems that we observed in regard to Example 2 may easily lead the unwary from a view of mathematics as a study heavily concerned with formal systems, back to the "vulgar formalist" view of mathematics as a particular formal system. If patterns of derivations in formal systems, and in classes of formal systems, can be modeled successfully by other formal systems, perhaps there is no real difference between the vulgar and refined versions of formalism. But, the vulgar view leads to an infinite regress of modeling systems in one another, explained very nicely by Lewis Carroll in "What the tortoise said to Achilles." As a last defense, the vulgar formalist might point out that we may choose an individual formal system, such as the Combinator Calculus, that is capable of modeling the behavior of *every* other formal system. Perhaps this cuts off the infinite regress, by taking the Combinator Calculus as the single source of all formal description (I can't resist comparing it to the bottom turtle in the famous story of the world being supported by "turtles all the way down."). No, there is still an infinite regress of *modeling* steps. The Combinator Calculus provides a single language in which each of the modeling steps may be described, but it does not cure the infinitude of steps.

**Characterizing formal systems in general.** I have avoided anything like a general definition of formal system until now, because I think that it's hard to appreciate the components of such a definition without some previous observation of formal systems and their uses.

**Definition 1 (Formal system)**

- A *formal alphabet* is a finite set of discrete symbols, reliably distinguishable from one another.

- A *formal language* is a set of finite discrete arrangements of symbols from a given formal alphabet, with a clear and unambiguous characterization of the relevant qualities of an arrangement.

- A *formal system* is a system of rules for deriving some of the arrangements of symbols from a given formal language, with a clear and unambiguous characterization of the manipulations that are and are not allowed as steps in such derivations.

**End Definition 1** □

Mathematical treatments of formal systems tend to use particular conventional styles of presentation of formal alphabets, languages, and rules, leading to the illusion that formality consists of following such a style. Following accepted mathematical style is not necessary for the presentation of a formal system—it is certainly feasible to establish other conventions that are equally as clear and unambiguous. Nor is it sufficient—it requires some pre-indoctrination in the rules and meaning of the conventional style. But, the use of mathematically conventional style is generally efficient, since the pre-indoctrination is reusable for presenting lots of different formal systems. What is crucial, though, is the clear and unambiguous mutual understanding of all parties to a discussion about a formal system. Often, that understanding is best established by some questions and answers with examples. In practice, we can be highly successful in establishing mutual understanding of the rules of a formal system.

Finally, why do I present a study of formal systems as part of my program to apply ideas from computer science to other disciplines? A derivation in a formal system is precisely the same thing as a computation. Computational systems and formal systems are the same things, and studies of the two differ in the qualities that one is motivated to consider, rather than the objects under study. Alan Turing's "On computable numbers, with an application to the Entscheidungsproblem," and Emil Post's "Finite combinatory processes" explore the human capacity for carrying out computation in a pseudophysical style that complements nicely Curry's more methodological study.

# 4 The Origin of Formal Systems

The concept of formal systems was not designed by a committee to satisfy a research contract—it arose from centuries of work by mathematicians and philosophers who were addressing individual problems with formal components, rather than seeking to design a general formal method, and it was characterized rigorously only in the early twentieth century. Nonetheless, I propose that we understand formal systems as a concept that is adapted very precisely to meet sensible goals, even though its design was much more evolutionary than conscious. The value of formal systems can best be appreciated by considering how a fictitious conceptual engineer might have designed them.

The earliest useful formal system that I can identify is the system of nonnegative integers, also called the counting numbers. We are so familiar with the counting numbers that many people sense that they are real objective physical objects. I find that numbers are real and objective, but not physical. The physical view of numbers appears to be founded on experiences with collections of discrete objects, such as pebbles placed in a bowl. Arguably $3 + 1 = 4$ is an observable physical fact, because the result of placing 3 pebbles in a bowl, then adding 1 more pebble, yields 4 pebbles in the bowl. But, the physical observations that support the concept of counting numbers must be filtered by some previous understanding of number. When we place 3 drops of water in a bowl, then add 1 more drop of water, we find 1 drop of water in the bowl. This experience does not lead us to question the validity of $3 + 1 = 4$, nor even to view it as a mere approximation of the truth about numbers. Rather, it leads us to conclude that the number of distinguishable drops of water in a bowl does not follow the rules of addition of counting numbers.

If the counting numbers are not physical observables, are they merely psychological ephemera, or even illusions? I think not. The actual practice of arithmetic in the world suggests that we understand the counting numbers very well as essentially conceptual constructs in a formal system (Curry called them *formal objects*, or *obs* for short). A bit more carefully, the system of counting numbers represents the common properties of a large class of formal systems, each of which contains objects representing the numbers and derivations corresponding to arithmetic calculations with numbers. $3 + 1 = 4$, then, is an observation about the qualities of the formal systems of counting numbers. We may convince ourselves very reliably that $3+1 = 4$ by choosing a particular formal system, and carrying out a derivation similar to the one in Example 1. In fact, the practice of placing pebbles in a bowl can be understood as a presentation of a formal system for the counting numbers—perhaps as an approximate presentation, since the bowl has a limited capacity.

The usefulness of formal systems (and the effectiveness of mathematics) derives from the fact that many natural phenomena are exact or approximate presentations of formal systems. If we can identify such formal systems and discover their properties, we can characterize some of the properties of the corresponding natural phenomena. Since formal systems may use any precisely and unambiguously characterized notion of pattern, we have a chance to apply formal systems to all natural phenomena in which we recognize such patterns. Reasoning about formal systems is useful because it often tells us that the presence of one pattern that we have observed entails the presence of another pattern that we have not noticed. For example, the pattern of definition of numerical addition in terms of successor entails that the result of addition is independent of order. This famous pattern is the *commutative property* of addition, written $x + y = y + x$ in algebraic texts.

Our understanding even of primitive propositions, such as $3 + 1 = 4$, involves the recognition of patterns among formal systems, rather than the mere exercise of a single formal system. Nobody really endorses $3+1 = 4$ just because of the specific derivation in Example 1. Rather, we recognize that a large class of formal systems exhibit certain qualities in common, which we regard as the form of numerical arithmetic, and that all of these systems share a pattern which we describe by $3+1 = 4$. The derivation in Example 1 is simultaneously an example of the $3 + 1 = 4$ pattern in a particular formal system, and a formal presentation of the pattern of reasoning that we follow in recognizing the $3 + 1 = 4$ pattern as an unavoidable consequence of the qualities shared by all of the formal presentations of numerical arithmetic.

If they are mental and abstract, rather than physical, how can formal systems be real and objective? They were evolved to be so, by eliminating from mental processes all the parts that we cannot agree on reliably. Suppose that we tasked our fictitious conceptual engineer to produce a method of reasoning, and communicating the results of reasoning, with the greatest possible capability for

- minimizing the physical cost of reasoning and communication, and
- maximizing objective certainty in the conclusions derived by reasoning.

If he were sufficiently brilliant, our engineer might consider that manipulation of symbols can be made physically very cheap, because we are always at liberty to substitute a lighter weight symbol for a heavier one, as long as all parties to a discussion recognize the same selection of symbols. Next, the rules for manipulating symbols should be based on the forms of arrays of those symbols, rather than a particular assigned meaning, in order to allow reasoners to maximize objective certainty. If the rules for manipulating symbols depend on their meanings, then our certainty about the correctness of manipulations is limited by our certainty about the behavior of the things that they refer to. We would like to use reasoning to illuminate qualities of things about which we are initially quite uncertain. So, we choose rules that refer only to the forms of arrays of symbols, and then we are at liberty to present the arrays and the rules in ways that we have discovered in practice to be thoroughly clear and unambiguous. It appears that our conceptual engineer has just designed for us the highly successful method of reasoning with formal systems.

Those with a taste for ontological studies may find my treatment of formal systems disturbingly circular. Instead of characterizing the essence of formal systems, and showing why that essence leads to certainty about the results of reasoning, I suggest that formal systems are the systems that we design for ourselves by refusing to deal with material about which we are uncertain. I concede the circularity, but I think that it represents the best way of understanding formal systems. They are social objects, designed for the purpose of communication. Because the design is so successful, they acquire an a posteriori air of necessity. In the next section, I trace the quest for certainty through the work of Descartes and Hilbert, and I think that this circular and somewhat negative view is at least highly consistent with their published ideas.

Another famously successful early example of a formal system is Euclidean geometry. Practitioners of geometry did not show clear understanding of the formal quality that characterized their work, but they had a strong intuitive sense that this work was reliable in a way quite different from other philosophical inquiries. Oddly, geometry was held for centuries to be an exact description of the layout of the physical universe, and the coexistence of formal certainty with physical factuality puzzled thinkers, such as Emmanuel Kant. Now we understand very well the sense in which geometry is a formal system and the resulting certainty in its conclusions, but we no longer believe that it describes the physical universe exactly.

## 5 Descartes' and Hilbert's Quests for Certainty

**Descartes' *Discourse* as a task description for the design of formal systems.** In his *Discourse on the Method of Rightly Conducting the Reason, and Seeking Truth in the Sciences*, René Descartes sought "clear and certain knowledge of all that is useful in life." He described a method that he expected to lead to such "clear and certain knowledge," given enough time and careful effort. Although it does not explicitly distinguish formal from contentual reasoning, the *Discourse* may be understood as containing a task description for our fictitious engineer, and substantial good advice to lead her toward her brilliant design.

Although he never showed an explicit awareness of something like formal systems, Descartes recognized mathematics, and geometry in particular, as a domain in which certainty had been achieved.

> I was especially delighted with the mathematics, on account of the certitude and evidence of their reasonings; but I had not as yet a precise knowledge of their true use; and thinking that they but contributed to the advancement of the mechanical arts, I was astonished that foundations, so strong and solid, should have had no loftier superstructure reared on them. On the other hand, I compared the disquisitions of the ancient Moralists to very towering and magnificent palaces with no better foundation

> than sand and mud: they laud the virtues very highly, and exhibit them as estimable far above anything on earth; but they give us no adequate criterion of virtue, and frequently that which they designate with so fine a name is but apathy, or pride, or despair, or parricide.

In the description of his method, Descartes does not explicitly mention the need for objective judgments, nor the need to communicate reasoning to others. But, in the supporting material he claims that others can follow his own method, and in particular that even children may follow the rules of arithmetic.

> The child, for example, who has been instructed in the elements of Arithmetic, and has made a particular addition, according to rule, may be assured that he has found, with respect to the sum of the numbers before him, all that in this instance is within the reach of human genius. Now, in conclusion, the Method which teaches adherence to the true order, and an exact enumeration of all the conditions of the thing sought, includes all that gives certitude to the rules of Arithmetic.

Here Descartes mentions his admiration of geometry.

> The long chains of simple and easy reasonings by means of which geometers are accustomed to reach the conclusions of their most difficult demonstrations, had led me to imagine that all things, to the knowledge of which man is competent, are mutually connected in the same way, and that there is nothing so far removed from us as to be beyond our reach, or so hidden that we cannot discover it, provided only we abstain from accepting the false for the true, and always preserve in our thoughts the order necessary for the deduction of one truth from another.

And here he claims that the conclusions of geometry are certain precisely because the method of geometry follows his rules.

> I was disposed straightway to search for other truths; and when I had represented to myself the object of the geometers, which I conceived to be a continuous body, or a space indefinitely extended in length, breadth, and height or depth, divisible into

> divers parts which admit of different figures and sizes, and of being moved or transposed in all manner of ways, (for all this the geometers suppose to be in the object they contemplate,) I went over some of their simplest demonstrations. And, in the first place, I observed, that the great certitude which by common consent is accorded to these demonstrations, is founded solely upon this, that they are clearly conceived in accordance with the rules I have already laid down.

Although Descartes probably did not understand completely the sense in which geometry is a formal system, it is telling that he picks a formal system as the only clear example of the success of his own method. It is reasonable to conclude that Descartes intends his method, at least when it is applied to mathematical topics, to coincide with the method of reasoning with formal systems.

I must concede that Descartes' conception of the certainty of geometry was not the same as the modern one. He surely believed that the postulates of Euclidean geometry represent certain and axiomatic knowledge about the physical universe. Relativistic physics does not support the correctness, much less the certainty, of the parallel postulate, and quantum physics suggests fundamental flaws in the concepts through which the postulates are expressed. The certainty in geometry now appears only to be that the postulates entail a rich set of additional information about Euclidean configurations. I prefer to imagine that Descartes had a correct, but vague, intuition about the real certainty in geometric deductions, and was wrong only about the certainty of its postulates, rather than to imagine that he had something radically different from a formal system of geometry in mind.

Now, look at Descartes' description of his method. This is important enough to quote in full, and in several versions. I give, first, the English translation by John Veitch, then the original French written by Descartes, then the Latin translation by Descartes' friend Etienne De Courcelles, which was revised by Descartes himself, and finally an English translation by Laurence J. Lafleur that combines the French version with the Latin: portions in parentheses are from the Latin only, in square brackets French only, in other portions the French and Latin agree.

Descartes' first rule states the requirement of certainty in the results of reasoning, although it does not mention objectivity or agreement between different reasoners. Its style is consistent with my suggestion that we achieve certainty by rejecting all material that is unclear or ambiguous.

> The *first* was never to accept anything for true which I did not clearly know to be such; that is to say, carefully to avoid precipitancy and prejudice, and to comprise nothing more in my judgement than what was presented to my mind so clearly and distinctly as to exclude all ground of doubt.
>
> Le premier était de ne recevoir jamais aucune chose pour vraie que je ne la connusse évidemment être telle: c'est-à-dire d'éviter soigneusement la précipitation et la prévention; et de ne comprendre rien de plus en mes jugements, que ce qui se présenterait si clairement et si distinctement à mon esprit, que je n'eusse aucune occasion de le mettre en doute.
>
> Primum erat, ut nihil unquam veluti verum admitterem nisi quod certò & evidenter verum esse cognoscerem; hoc est, ut omnem præcipitantiam atque anticipationem in judicando diligentissimè vitarem; nihilque amplius conclusione complecterer quàm quod tam clarè & distinctè rationi meæ pateret, ut nullo modo in dubium possem revocare.
>
> The first rule was never to accept anything as true unless I recognized it to be (certainly and) evidently such: that is, carefully to avoid (all) precipitation and prejudgment, and to include nothing in my conclusions unless it presented itself so clearly and distinctly to my mind that there was no (reason [or) occasion] to doubt it. The second, to divide each of the difficulties under examination into as many parts as possible, and as might be necessary for its adequate solution.

The second rule does not connect directly to my discussion above, but it resembles the reasoning used by Turing and Post in their computational versions of formal systems, where they divided computational steps into pieces small enough that there could be no doubt about their correctness.

> The *second*, to divide each of the difficulties under examination into as many parts as possible, and as might be necessary for its adequate solution.
>
> Le second, de diviser chacune des difficultés que j'examinerais, en autant de parcelles qu'il se pourrait et qu'il serait requis pour les mieux résoudre.

> Alterum, ut difficultates, quas essem examinaturus, in tot partes dividerem, quot expediret ad illas commodiùs resolvendas.
>
> The second was to divide each of the difficulties which I encountered into as many parts as possible, and as might be required for an easier solution.

The third rule does not make a definitive connection to formal systems, but it is certainly consistent with the progression, in formal systems of logical reasoning, from simple postulates to more subtle and complex theorems.

> The *third*, to conduct my thoughts in such order that, by commencing with objects the simplest and easiest to know, I might ascend by little and little, and, as it were, step by step, to the knowledge of the more complex; assigning in thought a certain order even to those objects which in their own nature do not stand in a relation of antecedence and sequence.
>
> Le troisième, de conduire par ordre mes pensées, en commençant par les objets les plus simples et les plus aisés à connaître, pour monter peu à peu, comme par degrés jusqu'à la connaissance des plus composés, et supposant même de l'ordre entre ceux qui ne se précèdent point naturellement les uns les autres.
>
> Tertium ut cogitationes omnes, quas veritati quærendæ impenderem, certo semper ordine promoverem: incipiendo scilicet à rebus simplicissimis & cognitu facillimis, ut paulatim, & quasi per gradus, ad difficiliorum & magis compositarum cognitionem ascenderem; in aliquem etiam ordinem illas mente disponendo, quæ se mutuò ex natura sua non præcedunt.
>
> The third was to think in an orderly fashion (when concerned with the search for truth), beginning with the things which were simplest and easiest to understand, and gradually and by degrees reaching toward more complex knowledge, even treating, as though ordered, materials which were not necessarily so.

The fourth rule also expresses a necessary, but not definitive, quality of formal systems.

> And the *last*, in every case to make enumerations so complete, and reviews so general, that I might be assured that nothing was omitted.

> Et le dernier, de faire partout des dénombrements si entiers, et des revues si générales que je fusse assuré de ne rien omettre.
>
> Ac postremum, ut tum in quærendis mediis, tum in difficultatum partibus percurrendis, tam perfectè singula enumerarem & ad omnia circumspicerem, ut nihil à me omitti essem certus.
>
> The last was (, both in the process of searching and in reviewing when in difficulty,) always to make enumerations so complete, and reviews so general, that I would be certain that nothing was omitted.

One cannot discuss the *Discourse* without mentioning its most famous sentence: "Je pense, donc je suis," "Cogito ergo sum," "I think, therefore I am." I regard this fascinating sentence as an unsuccessful attempt to apply the four rules. In particular, it represents an attempt to provide a new postulate about existence with the same sort of certainty that Descartes associated, incorrectly, with the postulates of geometry. While formal systems provide very strong certainty about their derivations, his proposed extension of certain knowledge to fundamental postulates is probably impossible to achieve.

**Hilbert restricted Descartes' program to mathematics.** In "The foundations of mathematics," and "On the infinite," David Hilbert proposed a program to "recast mathematical definitions and inferences in such a way that they are unshakable." Hilbert relied explicitly on formal systems as the tool for achieving unshakable certainty regarding all of mathematics. Although he did not refer explicitly to Descartes, we may read the first part of Hilbert's program as the restriction of Descartes' program to mathematics, instead of "all that is useful in life."

In these lectures, Hilbert repeatedly uses the German word "inhaltlich." The word has no precise English counterpart in common usage. I have taken Stefan Bauer-Mengelberg's translation to "contentual," which I understand to mean *referring to the content or meaning of assertions or formulae*. "Contentual" is a sensible opposite to "formal." Curry translated "inhaltlich" as "contensive." It is sometimes rendered as "intuitive," which I find quite misleading. It seems clear that Hilbert is distinguishing between form and content—one may employ intuition in dealing with either of these. He refers at least once to the application of "perceptual intuition" to forms.

Hilbert's program has two major steps:

1. to recast all of mathematics within one formal system of reasoning, and

2. to prove, using only elementary reasoning about finite objects, the consistency of that formal system.

Step one may be understood as the restriction of Descartes' program to mathematics. If step one were successful, then we would have a uniform formal mechanism for deriving all mathematical truths, but we would still take time to produce each individual truth. We would enjoy thorough certainty that our derivations were correct in terms of the rules of the formal system, but not that the rules that themselves were appropriate. Step two gives primacy to the generation of one particular truth intended to secure the correctness of step one against every conceivable challenge. Hilbert was reacting in particular to the discovery of contradictions in basic set theory and the study of infinitesimals in the differential and integral calculus, which he considered as mathematical catastrophes. Although step two holds the key to Hilbert's motivation, I am concerned here with step one.

Hilbert expresses his intention to achieve certainty through formal systems quite explicitly.

> I should like to eliminate once and for all the questions regarding the foundations of mathematics, in the form in which they are now posed, by turning every mathematical proposition into a formula that can be concretely exhibited and strictly derived, thus recasting mathematical definitions and inferences in such a way that they are unshakable and yet provide an adequate picture of the whole science.
>
> In my theory contentual inference is replaced by manipulation of signs according to rules; in this way the axiomatic method attains that reliability and perfection that it can and must reach if it is to become the basic instrument of all theoretical research.
>
> [My theory's] aim is to endow mathematical method with ... definitive reliability.
>
> It is necessary to formalize the logical operations and also the mathematical proofs themselves.

> A formalized proof, like a numeral, is a concrete and surveyable object.

In mathematical practice, concepts of infinite objects are the ones that seem to call on questionable contentual intuitions, so Hilbert is particularly concerned with formalizing those concepts.

> Modes of inference employing the infinite must be replaced generally by finite processes that have precisely the same results.

Although Hilbert insists on formal systems as the only sources of certainty in mathematical inference, he appeals to direct contentual intuition for the truth of basic numerical equations, such as $3 + 1 = 4$.

> We recognize that we can obtain and prove [numerical] truths through contentual intuitive considerations.

I claim that our certainty about integer arithmetic derives from the essentially formal nature of the numbers (that is, the content of integer arithmetic is a particular sort of form). Hilbert insists on a distinction between the contentual appreciation of integer arithmetic and the formal description of other concepts in mathematics (he calls them "ideal" concepts, to indicate that they are not significant in themselves, but only as ways of organizing arithmetic truths). But, the certainty of integer arithmetic derives from our ability to check each formula by computation, which is just derivation in a formal system. I think it is fair to say that both the arithmetic ground of Hilbert's mathematics and its "ideal" superstructure are essentially formal systems, but that he considers the formal systems of arithmetic to be uniquely chosen for some prior reasons, while he regards the rules for reasoning about ideals as products of a less constrained design. In my view, the content of integer arithmetic is a sort of form. Hilbert chooses to emphasize its status as content, largely to combat the criticism of mathematics as merely a game played with symbols. But, our certainty about integer arithmetic derives from its formal nature.

Hilbert treats the concept of formal systems as a pre-existing tool to be used in his work. He mentions some key qualities of formal systems, but he does not inquire explicitly into the origin of formal systems and the sources of certainty.

> If logical inference is to be reliable, it must be possible to survey [mathematical] objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that neither can be reduced to anything else nor requires reduction.

Some passages at least suggest that we may regard formal systems as the natural outcome of design requirements. The phrase below, "according to the conception we have adopted," seems to acknowledge that formalisms are objective partly because we adopt precisely those formal distinctions that all parties agree to recognize.

> In mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable.

Hilbert's comparison of his approach to the foundations of mathematics to other proposals is consistent with the idea of formal systems as engineered concepts.

> Mathematics is a presuppositionless science. To found it I do not need God, as does Kronecker, or the assumption of a special faculty of our understanding attuned to the principle of mathematical induction, as does Poincaré, or the primal intuition of Brouwer, or, finally, as do Russell and Whitehead, axioms of infinity, reducibility, or completeness, which in fact are actual, contentual assumption that cannot be compensated for by consistency proofs.

A final passage acknowledges the importance of universal agreement, although it does not explain how formal systems lead to such agreement.

> Mathematics in a certain sense develops into a tribunal of arbitration, a supreme court that will decide questions of principle—and on such a concrete basis that universal agreement must be attainable and all assertions can be verified.

# 6 The Strength and Scope of Formal Certainty

**How certain are formal derivations?** Not *absolutely* certain, since they depend on consensus regarding formal distinctions, and the correct perception of those formal distinctions through whatever senses we choose for their presentation. I find this not very disturbing, and doubt the possibility of absolute certainty about anything. The correctness of formal derivations is at least as certain as primitive sensual observations, such as *the sky is blue* and *the sun rose this morning.* They are more robust, since we are at liberty to repeat the verification of a derivation to the limits of our attention and tolerance for tedium, and to recast the presentation of symbols whenever we notice a potential for error or ambiguity. Although not absolute, formal derivations arguably enjoy the highest degree of objective certainty that is attainable by rational intelligence.

The strength of certainty is not purely quantitative—there are different sorts of certainty. When we stand on Gibraltar, we feel an a priori sort of certainty in the independent quality of that rock as a support for our feet. Certainty in the derivation of formal systems is a more social sort of certainty. It shares some of the qualities of our certainty in an automobile that is warranted by a reliable firm. We are confident, but not that the auto will always function perfectly. Rather, we are confident that we can recognize deviations, and adjust the machine, with occasional appeal to the maker, so that it eventually gets us where we want to go. Similarly, we are certain about derivations in formal systems because we can detect errors, and we can refine our physical presentations of arrays of symbols to overcome momentary confusion and ambiguity. Because of the extreme efficiency and malleability of the basic symbolic materials underlying formal systems, the degree of certainty in final success is much stronger than the degree of certainty in even the best engineered automobiles.

**What assertions are we so certain about?** Strictly, a formal system only gives us strong certainty that certain derivations do or do not follow the rules of the system. They do not and cannot provide certainty that particular natural phenomena, such as the configuration of paths followed by particles of light, follow precisely the rules of a formal system, such as Euclidean or non-Euclidean geometry. But, the scope of formally derived certainty is much more valuable practically than this mere certainty relative to the rules suggests to pessimists. Reasoning about formal systems can give us extremely high certainty that the presence of one formal pattern entails the presence of another. Since our observations of the universe abound in, and arguably consist entirely of, recognitions of formal patterns, the actual effectiveness of formal systems is substantial, and not at all unreasonable.

**What are the formal limits on formal studies?** Gödel's famous incompleteness theorem shows the first step in Hilbert's program to be inherently impossible to achieve. No single formal system can derive all of the truths of integer number theory. If we accept that Descartes' program contains the first step in Hilbert's, then his program is also inherently impossible. The second step in Hilbert's program depends on the first. But, if we choose a single formal system that is sufficient for some part of mathematical practice, there is still value in proving the consistency of that limited system. Gödel also showed that consistency of one formal system requires reasoning that is in some technical sense too powerful to be carried out in the system under study. It is natural to conclude that Hilbert's second step is impossible, even accepting a limited accomplishment of the first step. Takeuti pointed out that the technical power of a system involved in Gödel's theorem is not necessarily connected to the ontological level of our confidence in the system. So, it makes sense to prove the consistency of a formal system using rules of reasoning that are technically more powerful, but intuitively more secure, than those of the system under investigation. The practical impact of this approach to Hilbert's second step has only been partially explored.

# 7 More to Think About

- For a deeper look at many of the issues introduced in this article, here are some further readings from the bibliography.
  - Haskell B. Curry explores formalism as the content of mathematics in *Outlines of a Formalist Philosophy of Mathematics* and *Combinatory Logic, Volumes I and II.* The two books on *Combinatory Logic* are mostly full of mathematical technicalities, but the first chapter of each discusses the philosophy of formalism and its relation to mathematics. In particular, Chapter 11, the first of Volume

II, reacts to misunderstanding of the presentation in Chapter 1 of Volume I, perhaps by the same vulgar formalists who annoyed Mac Lane.

  - A. M. Turing's "On computable numbers" and E. Post's "Finite combinatory processes" explore the way that the structure of computation derives from the physical processes involved in computing by humans.
  - R. W. Hamming and E. P. Wigner, in "The unreasonable effectiveness of mathematics ... ," explore the practice of pure and applied mathematics, and describe some phenomena that support the notion of mathematics as the result of conceptual engineering.
  - S. Mac Lane, in *Mathematics, form and function*, gives the most thorough and accurate treatment that I have seen of the rich entanglement of mathematics with formal systems at a number of levels.
  - I annotated Descartes' *Discourse*, and the two lectures by Hilbert, in somewhat more detail for a college course at the University of Iowa. You may view the annotations on the World Wide Web at *http://www.cs.uchicago.edu/~odonnell/OData/Courses/22C:096/Lecture_notes/contents.html*.

- Investigate the actual evolution of systems of symbols, using linguistic and psychological methods to illuminate the weeding out of ambiguity in recognizing symbols and their arrangements (not the same as ambiguity in their meanings). Investigate the interaction with efficiency of presentation.

- Find examples for early successful uses of formal systems, besides integer arithmetic and geometry. Seek, in particular, more primitive systems that may not have been associated consciously with mathematics.

- Find examples for natural phenomena with formal properties as reliable as the numerical properties of pebbles in a bowl. Notice limits on the exactness of even the best of these formal descriptions of nature (for example, limits on the number of pebbles that might ever be contained in a bowl).

- Trace the changing attitude toward formal geometry over the centuries.
- Follow the precursors of, and explicit references to, computation and formal systems through the works of other philosophers, particularly Emmanuel Kant.
- Investigate the importance of the reflexivity of formal systems—the formal study of formalism. Does it contribute to the certainty achieved by such systems? How does it affect the character of formal studies?
- Present the story of Hilbert's program as a tragedy in the formal sense defined by Aristotle.
- Analyze computational systems as systems designed to have objectively communicable results. Many qualities of computational systems may derive from the limitations of robust communication, due to the information bottleneck imposed by language. There might be a new type of evidence for the Church-Turing thesis here (thesis: Turing machines, Combinatory Calculus, and some other known formal systems characterize precisely the computable functions).

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