**Question 1** Prove that if two opposite corners of the chessboard are removed, the board cannot be covered by dominoes.

**Question 2** Consider an $m \times n$ grid.

1. Count the number of non-empty subrectangles.
2. Count the number of paths that start at bottom-left corner and go to top-right corner and can only move up or (inclusive) right.

**Question 3** Sort the following functions in a non-decreasing order of asymptotic growth:

$n^3 \log n^2 \quad 4 \log n \quad n^4 \quad n! \quad \binom{n}{4}$

where log is to the base 2. Specify and prove the asymptotic relationship (little-“oh”, little-“omega”, big-“theta”) between the consecutive functions in your order.

**Question 4** How many ways are there to climb a staircase with $n$ steps if we can climb 1 stair in 2 different ways or jump 2 stairs in 3 different ways.

**Question 5** A drunk has $n$ keys and only one will open the door. He tries keys at random. Under each model below, what is the expected number of selections until he opens the door?
1. He selects keys in a random order (without replacement) until one works.

2. After each mistake, he replaces the key and selects randomly again.

**Question 6** Each of \( n \) hunters selects a rabbit at random from a group of \( n \) rabbits, aims a gun at it, and then all the hunters shoot at once. Let \( X \) be a random variable that is equal to the number of rabbits that survive. What is \( E(X) \)?

**Question 7** How many ways are there to place 10 distinct people within 3 distinct rooms? How many ways are there to place 10 distinct people within 3 distinct rooms so that each room receives at least one person?

**Question 8** Consider the following variant on the game of life on the \( 8 \times 8 \) chessboard. Initially, some of the cells are alive. At each time step a new cell is born in an empty cell if two of the neighbors are alive. Note: neighbors must share an edge, so diagonal cells are not neighbors. Each alive cell leaves forever. What is the minimum initial population so that the life eventually spreads to the entire board? Show that 8 is sufficient and that 7 is not.

**Question 9** Prove that every polynomial \( f(x) \neq 0 \) has a multiple \( g(x) = f(x)h(x) \neq 0 \) in which every exponent is prime. I.e., \( g \) is of the form \( g(x) = \sum_{p\text{-prime}} a_p x^p \).