Flag Algebras: an Interim Report

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To the memory of my mother,
Ludmila Alexeevna Razborova

Abstract

For the most part, this article is a survey of concrete results in extremal combinatorics obtained with the method of flag algebras. But our survey is also preceded, interleaved and concluded with a few general digressions about the method itself. Also, instead of giving a plain and unannotated list of results, we try to divide our account into several connected stories that often include historical background, motivations and results obtained with the help of methods other than flag algebras.

A foreword

When I was asked by the organizers to contribute something on flag algebras, I was a bit uncertain at first. The reasons will become clear from the text below, but a two-sentence summary is this. In just a few recent years we have witnessed a tremendous explosion of activity in this area, and the explosion is still ongoing. It does not look (at least to me) quite consistent with the inevitable stamp of finality a full-fledged survey is supposed to convey.

As a consequence, this contribution has a very clear flavor of an accounting book. I will try my best to summarize in Section 3, in a categorized and

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annotated form, *concrete* results in extremal combinatorics obtained with the method of flag algebras so far. Or, in other words, where do we stand now, in February of 2013.

That said, I still feel obliged to say at least a few general words about the method itself, and this is where we begin. This introductory part is rather loose and informal, and a disinterested reader may proceed directly to Section 2.

1. The method

The theory of flag algebras is supposed to treat in an entirely uniform way all classes of combinatorial structures $\mathcal{C}$ that possess the *hereditary property*: any subset of vertices of a structure from $\mathcal{C}$ gives rise to another ("induced") structure in $\mathcal{C}$. A precise definition at the appropriate level of generality is best given in logical terms [Raz07, §2], but for the purposes of this text we can safely assume that $\mathcal{C}$ is the class of either ordinary simple graphs or r-uniform hypergraphs (*r-graphs*) or oriented graphs (*orgraphs*). In this section the specific choice of the class $\mathcal{C}$ is almost never important, and for simplicity we will use the word "graph" cumulatively.

The main quantity studied in the part of extremal combinatorics that is amenable to the method of flag algebras is the number $i(H,G)$ of induced copies (up to automorphisms of H) of a graph H in a larger graph G. One of the most basic paradigms underlying the theory of flag algebras tells us to normalize whenever possible so we immediately replace these numbers with the corresponding densities and let

$$p(H,G) \stackrel{\text{def}}{=} \binom{L}{\ell}^{-1} i(H,G)\ (L \stackrel{\text{def}}{=} |V(G)|,\ \ell \stackrel{\text{def}}{=} |V(H)|).$$

One useful interpretation is that $p(H,G)$ is the probability that a randomly chosen ℓ-subset of $V(G)$ induces a subgraph isomorphic to H [Raz07, §2.1].

In many contexts, notably in the theory of graph limits, researchers are often interested in the number of all copies, not necessarily induced, and sometimes also other variants. It turns out, however, that all these variants are essentially equivalent; let us review some simple formulas connecting different versions (see [Lov12, Chapter 5.2.2]) as we will occasionally need them below.

Let $\mathsf{ind}(H,G)$ be the number of induced embeddings $\alpha : V(H) \longrightarrow V(G)$, that is embeddings preserving both adjacency and non-adjacency. Denoting by

$$t_{\mathsf{ind}}(H,G) \stackrel{\text{def}}{=} \frac{\mathsf{ind}(H,G)}{L(L-1)\cdots(L-\ell+1)}$$

the corresponding density, we see that $\mathsf{ind}(H,G) = |\mathrm{Aut}(H)| \cdot i(H,G)$ and, hence,

$$t_{\mathsf{ind}}(H,G) = \frac{|\mathrm{Aut}(H)|}{\ell!} p(H,G).$$

$t_{\mathsf{inj}}(H,G)$ is defined similarly to $t_{\mathsf{ind}}(H,G)$ with the difference that now the embedding α need not necessarily be induced, i.e. it must respect adjacencies only. Clearly,

$$t_{\mathsf{inj}}(H,G) = \sum_{H' \supseteq H} t_{\mathsf{ind}}(H',G) = \frac{1}{\ell!} \sum_{H' \supseteq H} |\mathrm{Aut}(H')| p(H',G), \tag{1}$$

and the inverse formula is given by the Möbius transform (see [Lov12, (5.20)]):

$$t_{\mathsf{ind}}(H,G) = \sum_{H' \supseteq H} (-1)^{|E(H')|-|E(H)|} t_{\mathsf{inj}}(H',G). \tag{2}$$

Two more variants, homomorphism density $t(H,G)$ [Lov12] and strong homomorphism density [HHK+11, Section 2.3] are obtained from $t_{\mathsf{inj}}(H,G), t_{\mathsf{ind}}(H,G)$, respectively, by dropping the requirement that the mapping α must be injective, followed by an obvious re-normalization. They are related to each other via formulas completely analogous to (1), (2). There is no neat formula, however, relating "injective" densities $p(H,G), t_{\mathsf{ind}}(H,G), t_{\mathsf{inj}}(H,G)$ with their non-injective versions: any such formula must necessarily involve the number of vertices L which is grossly inconsistent with the philosophy of flag algebras. What is important, however, is that as $L \to \infty$, the difference between these two classes of measures becomes negligible (see e.g. [Lov12, (5.21)]).

A significant part of extremal combinatorics studies arithmetic and Boolean relations existing between the densities $p(H_1,G), \ldots, p(H_h,G)$ (or sometimes their equivalent versions $t_{\mathsf{ind}}(H_i,G), t_{\mathsf{inj}}(H_i,G)$) where $H_1, \ldots, H_h$ are small fixed templates, and G is an unknown graph. Sometimes problems of interest (like the Caccetta-Häggkvist conjecture that we will discuss in Section 3.3) also involve concepts like minimal/maximal degree; these fit into our framework with very minimal changes.

And the *asymptotic* extremal combinatorics additionally assumes that the size of G is very large, and thus these relations are to be satisfied only in the limit. More precisely, in every increasing sequence $G_1, G_2, \ldots G_n, \ldots$ of graphs we can by compactness choose a subsequence $G'_1, \ldots, G'_n, \ldots$ such that all h limits $\lim_{n\to\infty} p(H_\nu, G'_n)$ ($\nu \in [h]$) exist; denote them by $\phi(H_1), \ldots, \phi(H_h)$. The question is then re-phrased as follows: which properties should the tuple $(\phi(H_1), \ldots, \phi(H_h))$ satisfy?

The next observation is that by going to an infinite subsequence we can ensure that the limits $\phi(H) \stackrel{\text{def}}{=} \lim_{n\to\infty} p(H, G'_n)$ exist for *all* (countably many) graphs H, not only those we are actually interested in. This follows from Tychonoff's theorem on the compactness of products of compact sets (that in our particular case can be replaced by a simple diagonal argument). Such sequences are called *convergent*, and the function ϕ that maps isomorphism classes of finite graphs[1] is a paradigmical example of what in the theory of graph limits is called a simple graph parameter [Lov12, Chapter 4.1].

Convergent sequences of graphs $\{G_i\}$ and associated graph parameters ϕ make the main object of study in both theories: graph limits and flag algebras. From this point, however, they diverge significantly: a logician might have said that the theory of graph limits is semantical in its nature while flag algebras strongly focus on syntax. Indeed, a very substantial part of the theory of graph limits deals with the question of what is the *actual* limit object for a converging sequence of graphs and with studying its properties. This limit object was successfully described by Lovász and B. Szegedy for ordinary graphs (graphons, see [Lov12, Chapter 7]), by Elek and B. Szegedy for hypergraphs [Lov12, Chapter 23.3], and it looks as if a sort of a description is possible even for directed graphs [Lov12, Chapter 23.5].

The approach taken by flag algebras is on the contrary manifestly minimalistic which is dictated by the utilitarian purpose of the theory. Semantics is substantially demoted as not being very useful for proving new concrete results; one immediate advantage of this is that the theory can be applied to arbitrary combinatorial structures without any changes at all. Instead, it focusses on developing *syntactic* tools for proving universal statements about the quantities $\phi(H_1), \ldots, \phi(H_h)$ using more or less formal manipulations. A careful attention is paid to notational uniformity, simplicity and transparency: this is particularly important since, as the experience shows,

[1]It is perhaps a good time to remind that in this section we use the word "graph" in a broader sense that also includes hypergraphs, orgraphs etc.

the method begins to bring real fruit dangerously close to the region where it becomes unfeasible for purely computational reasons, see the discussion in [FRV13, Section 4.1]. Another characteristic feature of the method is its strong tendency to expose and exploit (usually simple) mathematical structure in an uniform way wherever it can be found. Besides obvious mathematical connections, this paradigm, somewhat surprisingly, has its own non-negligible utilitarian value. For example, it adds versatility to some existing packages for working with flag algebras that can be easily re-programmed to work with different types of combinatorial objects.

As we indicated at the beginning, this text is not intended to be an exposition of the method itself. Almost all necessary formalism can be found in the original paper [Raz07]; [Raz11b, Section 2.1.1] adds half a page of notation and definitions that are particularly useful when one has to work with several different types of combinatorial structures at once. An informal account can be found in [Kee11, Section 7], and almost every paper with concrete results surveyed below also strives to explain its own version of the formalism in its own way. But in the next section 3 we will use the distinction between "plain" (Cauchy-Schwarz) applications and those using more advanced concepts. So we conclude with a somewhat informal account of the fragment of the general theory that is necessary to understand this distinction. This part is similar to quantum graphs, graph algebras, reflection positivity etc. studied in the context of graph limits [Lov12, Part 2], but there are also important differences dictated, as almost everything else in flag algebras, by pragmatic purposes.

Let $\mathcal{M}$ be the set of all finite graphs up to an isomorphism, and $\mathbb{R}\mathcal{M}$ be the set of all their finite formal linear combinations with real coefficients. Then any graph parameter ϕ can be extended by linearity to a linear mapping $\mathbb{R}\mathcal{M} \to \mathbb{R}$ that we will also denote by the same letter ϕ. Graph parameters ϕ resulting from convergent sequences of graphs turn out to satisfy $\phi(f) = 0$ for certain elements $f \in \mathbb{R}\mathcal{M}$ expressing the most basic *chain rule* [Raz07, Lemma 2.2]. Factoring out by these relations, we obtain a linear space that is denoted by $\mathcal{A}^0$ (the meaning of the superscript 0 will become clear soon). It turns out that for every pair H_1, H_2 of graphs, $\phi(H_1)\phi(H_2)$ can be always expressed as $\phi(f)$ for an easily computable element $f \in \mathcal{A}^0$ not depending on ϕ. This allows us to endow $\mathcal{A}^0$ with the structure of an associative commutative algebra [Raz07, Lemma 2.4], and thus ϕ defines an algebra homomorphism from this algebra to the reals. It clearly satisfies $\phi(H) \geq 0$ for any graph H, and we let $\mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ denote the set of all algebra homomorphisms with the latter property.

One extremely important fact is that at this point our search for "generic", "logical" relations satisfied by all graph parameters resulting from convergent sequences is over. Namely, the "completeness theorem" ([Lov12, Theorem 11.52], [Raz07, Theorem 3.3]) states that *every* element $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ can be realized as a convergent sequence of graphs, and this allows us to focus on $\mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ as an *axiomatic* description of our main object of study. Of course, even under this view, the intended semantical interpretation is still indispensable for intuition and is occasionally used in arguments (see e.g. [Raz07, Theorem 4.3]).

The backbone of the theory is made by the real cone $\mathcal{C}^0_{\mathrm{sem}}$ consisting of all those f for which $\forall\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})(\phi(f) \geq 0)$, and what we refer to as "plain" Cauchy-Schwarz applications is just a systematic way of finding "interesting" elements in this cone by semi-definite programming. More specifically, all notions reviewed so far readily generalize to the relative framework in which a prescribed number of vertices k spanning a prescribed graph σ are labeled in all objects under consideration and are always required to be preserved [Raz07, §2.1]. σ itself is called a *type* [of size k], relativized graphs become *flags* [of type σ], and the relativized version $\mathcal{A}^\sigma$ is (finally!) called the *flag algebra*. For every $f \in \mathcal{A}^\sigma$ we clearly have $f^2 \in \mathcal{C}^\sigma_{\mathrm{sem}}$, and we also have a naturally defined *averaging* (or *label-erasing*) linear operator $[\![\cdot]\!]_\sigma : \mathcal{A}^\sigma \longrightarrow \mathcal{A}^0$ preserving the set of positive elements: $[\![\mathcal{C}^\sigma_{\mathrm{sem}}]\!]_\sigma \subseteq \mathcal{C}^0_{\mathrm{sem}}$ [Raz07, Theorem 3.1]. This already provides us with a supply of non-trivial elements in $\mathcal{C}^0_{\mathrm{sem}}$ of the form $[\![f^2]\!]_\sigma$ ($f \in \mathcal{A}^\sigma$), and we can also take their linear combinations with non-negative coefficients. The resulting set is a quadratic sub-cone $\mathcal{C}^0 \subseteq \mathcal{C}^0_{\mathrm{sem}}$ defined by positive semi-definite constraints. And when the size of all flags involved is bounded by a constant ℓ (in a "typical" plain application of the method ℓ varies between 4 and 6), the corresponding SDPs become finitely defined, and, what is even more important can be handled by the existing solvers[2] sufficiently well to actually solve problems. This is what we will refer to as the "plain" method, and in what follows we will use the word "plain" in this rather technical and exact sense.

The structure that can be extracted from the objects $\mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ is, however, much richer than that and includes other things like various algebra

[2] In my own work, I interchangeably use CSDP [Bor99] and SDPA `http://sdpa.sourceforge.net/`, and my special thanks go to their developers.

homomorphisms allowing us to move true statements around [Raz07, §2.3], ensembles of random homomorphisms extending a given one [Raz07, §3.2] or variational principles [Raz07, §4.3]. We can not go into details here, but in Section 3 we will sometimes mention these structures by name whenever they are used in the arguments.

What are the relations between the cone $\mathcal{C}^0_{\text{sem}}$ we are interested in and its approximation $\mathcal{C}^0$ corresponding to what we can *prove* using Cauchy-Schwarz arguments? Topologically, $\mathcal{C}^0$ is dense in $\mathcal{C}^0_{\text{sem}}$, and one does not even have to use quadratic relations for that. Namely, it is a simple consequence of the completeness result [Lov12, Theorem 11.52], [Raz07, Theorem 3.3] that the linear subcone in $\mathcal{C}^0$ consisting of non-negative linear combinations of flags is already dense in $\mathcal{C}^0_{\text{sem}}$.

In terms of *logical* complexity, however, the difference is huge. If we for simplicity focus on rational points in these cones, then the sub-cone $\mathcal{C}^0_\ell \subseteq \mathcal{C}^0$ consisting of all inequalities provable by using only ℓ-sized flags is decidable and hence $\mathcal{C}^0 = \bigcup_\ell \mathcal{C}^0_\ell$ is recursively enumerable. The fundamental result by H. Hatami and Norin [HN11] states that $\mathcal{C}^0_{\text{sem}}$ is not r.e. already for ordinary simple graphs. Informally, this means that *every* proof system that will try to generate true statements in the asymptotic extremal combinatorics will necessarily be incomplete. Very recently, Lovett, H. Hatami, P. Hatami and Norin have extended this result to the theory of 2-colored graphs with distinguishable parts.

Finally, the theory of flag algebras has not appeared overnight out of nowhere, it had many predecessors. First of all, most constructions and arguments are modeled after their discrete counterparts that have been used in extremal combinatorics for many decades. Next, one should definitely mention the method of *Lagrangians* [MS65] that was perhaps the first successful usage of analytical methods in the area. Quasi-random graphs [CGW89] are, in our language, devoted to the study of one specific and, arguably, the most natural element of $\text{Hom}^+(\mathcal{A}^0, \mathbb{R})$, and many central results and proofs there have a distinct syntactic flavor. Bondy [Bon97] used what we would now call "Cauchy-Schwarz calculus" in the specific context of the Caccetta-Häggkvist problem.

2. Notation

We review the main definitions for the case of simple r-uniform hypergraphs (r*-graphs* in what follows), where $r \geq 2$ is a fixed number. We will also be occasionally considering oriented graphs[3], but it is very straightforward how to adopt our definitions to that case.

2.1. Turán densities

For two r-graphs F and G, G is F*-free* if it does not contain (not necessarily induced) subgraphs isomorphic to F. Given a family $\mathcal{F}$ of r-graphs, G is $\mathcal{F}$*-free* if it is F-free for every $F \in \mathcal{F}$. Let $\mathrm{ex}_H(n;\mathcal{F})$ be the maximal possible number of induced copies of an r-graph H in an $\mathcal{F}$-free r-graph on n vertices and

$$\pi_H(\mathcal{F}) \stackrel{\mathrm{def}}{=} \lim_{n\to\infty} \frac{\mathrm{ex}_H(n;\mathcal{F})}{\binom{n}{|V(H)|}}.$$

In the language of flag algebras, $\pi_H(\mathcal{F})$ is the maximal possible value of $\phi(H)$, where the maximum is taken over all $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ for which $\phi(\widehat{F}) = 0$ whenever $\widehat{F}$ contains a spanning subgraph isomorphic to some $F \in \mathcal{F}$. We let

$$\pi(\mathcal{F}) \stackrel{\mathrm{def}}{=} \pi_{\{e\}}(\mathcal{F}),$$

where e is a single (hyper)edge. For better understanding the context of this survey, it is useful to recall that in the case of ordinary simple graphs the quantities $\pi(\mathcal{F})$ are completely described by the Erdős-Stone-Simonovits theorem [ES46, ES66]:

$$\pi(\mathcal{F}) = 1 - \frac{1}{r-1}, \qquad (3)$$

where $r \stackrel{\mathrm{def}}{=} \min\{\chi(G) \mid G \in \mathcal{F}\}$.

In order to cover more situations of interest, we define $\mathrm{ex}_{\min,H}(n;\mathcal{F})$, $\pi_{\min,H}(\mathcal{F})$ analogously to $\pi(\mathcal{F})$, but with the following two differences:

1. we are interested in the *minimal* possible number of induced copies of H;

2. r-graphs from $\mathcal{F}$ are forbidden only as *induced* subgraphs.

[3] That is, directed graphs without loops, parallel or anti-parallel edges. By analogy with the abbreviation "digraph", in this survey oriented graphs will be often called *orgraphs*.

Again, when H is a single (hyper)edge, $\mathrm{ex}_{\min,H}(n;\mathcal{F}), \pi_{\min,H}(\mathcal{F})$ are abbreviated to $\mathrm{ex}_{\min}(n;\mathcal{F}), \pi_{\min}(\mathcal{F})$. Very recently, Norin (personal communication) was able to give a nice and complete description of $\pi_{\min}(\mathcal{F})$ for the case of ordinary graphs. More generally, given a finite set $\mathcal{F}$ of graphs he fully describes the set $D(\mathcal{F}) \subseteq [0,1]$ consisting of those $x \in [0,1]$ for which there exists $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ with $\phi(F) = 0$ $(F \in \mathcal{F})$ and $\phi(e) = x$. The situation for 3-graphs is very different, and some related results will be thoroughly discussed in Section 3.4.

When $\mathcal{F} = \{F\}$ consists of a single graph, the quantities $\pi_H(F)$ and $\pi_{\min,H}(F)$ can be readily generalized to their relaxed versions when instead of forbidding copies of F entirely, we are interested in minimizing their number. For example, given $x \in [0,1]$, we let

$$g_H^F(x) \stackrel{\mathrm{def}}{=} \lim_{n\to\infty} \frac{\mathrm{ex}_{H,x}(n;\mathcal{F})}{\binom{n}{|V(F)|}}, \tag{4}$$

where $\mathrm{ex}_{H,x}(n;\mathcal{F})$ is the minimal possible density of copies of F in an r-graph on n vertices in which the density of (induced) copies of H is at least x. Thus, g_H^F is a non-decreasing function and $\pi_H(F)$ is the maximal x for which $g_H^F(x) = 0$.

2.2. Frequently used [or]graphs

K_ℓ is a clique on ℓ vertices, I_ℓ is an independent set on ℓ vertices, C_ℓ $[\vec{C}_\ell]$ is a non-oriented [oriented, respectively] cycle of length ℓ, and P_ℓ $[\vec{P}_\ell]$ is a non-oriented [oriented] path on ℓ vertices, i.e., of length $(\ell-1)$. $\vec{K}_{1,\ell}$ is the oriented star on $(\ell+1)$ vertices in which all edges are oriented from the center.

2.3. Frequently used hypergraphs

K_ℓ^r is a complete r-graph on ℓ vertices, and I_ℓ^r is an empty r-graph on ℓ vertices (thus, $K_\ell = K_\ell^2$ and $I_\ell = I_\ell^2$). J_k is the 3-graph on $(k+1)$ vertices consisting of all $\binom{k}{2}$ edges that contain a distinguished vertex v. G_ℓ is the uniquely defined 3-graph on 4 vertices with ℓ edges; thus, $G_4 = K_4^3$, and G_3 is often denoted by K_4^-. $\mathcal{C}_5$ is the 3-graph on 5 vertices with the edge set $\{(123),(234),(345),(451),(512)\}$. $F_{3,2}$ is the 3-graph, also on 5 vertices, with the edge set $\{(123),(145),(245),(345)\}$.

$\bar{F}$ is the edge-complement of a (hyper)graph F (on the same set of vertices). For a (hyper)graph F, $\lambda(F)$ is its *Lagrangian* defined as the maximal possible edge density of all weighted hypergraphs resulted from placing probability distributions on the vertices of F.

3. Results

In our survey of existing results, we are trying to group them into a few large groups centered either around a "big" problem or a reasonably broad topic. In all these cases the contribution made by flag algebras has been very substantial, but seldom it was exclusive. Therefore, we feel that our purpose will be served better if we give more coherent account by including, whenever appropriate, historical context, motivations, results proved by other methods etc.

3.1. Clique densities

In this section we consider only simple ordinary graphs, and we are interested in the functions $g_{K_p}^{K_r}$ (see (4)). The case $p = 2$ has received most attention, and we abbreviate

$$g_r(x) \stackrel{\text{def}}{=} g_{K_p}^{K_r}(x).$$

In words, $g_r(x)$ is the (asymptotically) minimal possible density of K_r in graphs with edge density $\geq x$.

The first general bound on $g_r(x)$ was proved by Goodman [Goo59]:

$$g_3(x) \geq x(2x-1); \tag{5}$$

in the framework of flag algebras his proof amounts to a one-line calculation [Raz07, Example 11]. This result was later re-discovered by Nordhaus and Stewart [NS63] who also conjectured that

$$g_3(x) \geq \frac{2}{3}(2x-1). \tag{6}$$

Goodman's bound (5) was extended to the case $r = 4$ by Moon and Moser [MM62] as follows:

$$g_4(x) \geq x(2x-1)(3x-2) \; (x \geq 2/3).$$

Following the pattern, they also stated without proof the natural generalization

$$g_r(x) \geq \prod_{i=1}^{r-1}(ix-(i-1)) \ \left(x \geq 1-\frac{1}{r-1}\right) \tag{7}$$

for an arbitrary r; a complete proof was later provided in [KN78, LS83].

Values of the form $x = 1-\frac{1}{t}$ are called *critical*. These are precisely edge densities of complete balanced t-partite graphs, and at critical values the right-hand side of (7) computes the densities of K_r in these graphs. Thus, the bound (7) (and its partial case (5)) is tight at the critical points $1-\frac{1}{t}$; the question is what is happening between them.

The bound (7) is convex. Let $\psi_r(x)$ be the piecewise linear function that is linear in every interval $\left[1-\frac{1}{t}, 1-\frac{1}{t+1}\right]$ and coincides with g_r at its ends. Then, by convexity, $\psi_r(x) \geq g_r(x)$ (note that in the interval $[1/2, 2/3]$ the bound conjectured in (6) is precisely $\psi_3(x)$). More generally, let $\psi_r^p(x)$ be the piecewise linear function that is linear in every interval $\left[g_p\left(1-\frac{1}{t}\right), g_p\left(1-\frac{1}{t+1}\right)\right]$ and coincides with $g_r\left(1-\frac{1}{t}\right), g_r\left(1-\frac{1}{t+1}\right)$ at its ends.

In the beautiful paper [Bol76], Bollobás proved that $\psi_r^p(x)$ still provides a lower bound on the function $g_{K_p}^{K_r}$:

$$g_{K_p}^{K_r}(x) \geq \psi_r^p(x). \tag{8}$$

A brief survey of these and related early developments can be found in [Bol75].

We are not aware of any improvements on Bollobás's bound (8) for $p > 2$ which, in our opinion, makes an interesting open problem. The follow-up research concentrated on computing the functions $g_r(x)$.

As for upper bounds, let us consider a complete $(t+1)$-partite graph in which t parts are of the same size while the remaining part is smaller. Given $x \in \left[1-\frac{1}{t}, 1-\frac{1}{t+1}\right]$, there exists an asymptotically unique graph in this class with the edge density x. Computing the density of K_r in it leads to the following (somewhat ugly) upper bound on $g_r(x)$:

$$\left.\begin{aligned} g_r(x) \leq \frac{(t-1)!}{(t-r+1)!(t(t+1))^{r-1}} \cdot \left(t-(r-1)\sqrt{t(t-x(t+1))}\right) \\ \cdot \left(t+\sqrt{t(t-x(t+1))}\right)^{r-1} \ \left(x \in \left[1-\frac{1}{t}, 1-\frac{1}{t+1}\right]\right). \end{aligned}\right\} \tag{9}$$

This bound is *concave* in the interval $\left[1-\frac{1}{t}, 1-\frac{1}{t+1}\right]$.

I was not able to trace the origin of the conjecture that the bound (9) is actually tight, but in explicit form it appears already in the paper [LS83] by Lovász and Simonovits. The same paper proved the conjecture in some subintervals of the form $\left[1-\frac{1}{t}, 1-\frac{1}{t}+\epsilon_{r,t}\right]$, where $\epsilon_{r,t}$ is a (very small) constant. The next development occurred in 1989 when Fisher [Fis89] proved[4] that (9) is tight for $r=3, t=2$.

And this is where flag algebras entered the stage. Firstly, Razborov [Raz07, §5] independently re-discovered Fisher's result. More generally, a relatively simple calculation [Raz08, (3.6)] shows that if the bound (9) is tight for some t and $r=4$, then it is also tight for the same value of t and $r=3$. Fisher's result follows immediately since (9) is tight when $t=2, r=4$ (both sides are zero).

Then, using much more involved flag-algebraic constructs and calculations, Razborov [Raz08] proved that the bound (9) is tight for $r=3$ and an arbitrary t. In the classification scheme outlined in Section 1, this proof is certainly not plain, and in fact it barely uses Cauchy-Schwarz at all. Instead, it significantly employs more elaborated parts of the theory like ensembles of random homomorphisms or variational principles; we can not go into further details here.

While the next two papers do not directly use the language of flag algebras (see, however, the discussion at the conclusion of Section 1 in [Rei12]), the proofs are still highly analytical. Nikiforov [Nik11] proved that (9) is tight for $r=4$ (and any t). And, finally, Reiher [Rei12] established the same for arbitrary r, t thus completing the quest for computing the function g_r itself. Let me, however, remind here again that no progress on the relative values $g_{K_p}^{K_r}$ for $p>2$ has apparently been made since Bollobás's seminal paper [Bol76].

As for exact bounds, infinite blow-ups in general provide a powerful tool for converting asymptotic results into exact ones. In our context (we will discuss one more case in Section 3.5) this simple idea immediately implies the bound

$$\mathrm{ex}_{e,x}(n; K_r) \geq \frac{n^r}{r!} g_r\left(\frac{2m}{n^2}\right) \tag{10}$$

[Raz08, Theorem 4.1]. Nikiforov [Nik11, Theorem 1.3] showed that it is

[4] Fisher's proof was incomplete as it implicitly used a fact about clique polynomials unknown at the time. This missing statement, however, was verified in 2000.

rather close to optimal.

Lovász and Simonovits [LS83] made several quite precise conjectures about the behavior of $\mathrm{ex}_{e,x}(n; K_r)$ and the corresponding extremal configurations, but these conjectures still remain unanswered. A partial progress toward them was made by Pikhurko and Razborov in [PR12]. Firstly, using a genuine flag-algebraic argument, they completely described the set $\Phi \subseteq \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ of all asymptotically extremal configurations, i.e., those ϕ for which $\phi(K_3) = g_3(\phi(K_2))$. Then, by standard techniques, [PR12] proved stability, i.e. that actual extremal configurations are $o(n^2)$-close to the conjectured ones in the *edit* distance. These are precisely the first two steps of the program that we will discuss in the next section 3.2. The third step, however (extracting an exact result from the stability version) is still missing. And for $r > 3$ nothing along these lines seems to be known at all.

In conclusion, let me note again that since [Bol76] and [LS83] all improvements have been very analytical in their nature. Proving comparable results with entirely combinatorial techniques remains an unanswered challenge.

3.2. Turán's tetrahedron problem

In this section we switch gears and work with 3-graphs. The value $\pi(K_\ell^r)$ is unknown for *any* pair $\ell > r > 2$, this is the famous Turán problem. More information on its history and state of the art can be found in the recent comprehensive survey [Kee11] (see also much older but still useful text [Sid95]). In this section we concentrate on the simplest case $r = 3,\ \ell = 4$, with a brief digression to the next one, $r = 3,\ \ell = 5$. $\pi(K_4^3) = 1 - \pi_{\min}(I_4^3)$, and it will be convenient to us (partially for historical reasons) to switch to this dual notation. Turán [Tur41] conjectured that $\pi_{\min}(I_4^3) = 4/9$, and this conjecture is sometimes called *Turán's (3,4)-problem* or *tetrahedron problem*. De Caen [Cae91], Giraud (unpublished) and Chung and Lu [CL99] proved increasingly stronger lower bounds on $\pi_{\min}(I_4^3)$, the latter being of the form $\pi_{\min}(I_4^3) \geq \frac{9-\sqrt{17}}{12} \geq 0.406407$.

A plain (remember that we use this word in a technical sense) flag-algebraic calculation leads to the numerical bound

$$\pi_{\min}(I_4^3) \geq 0.438334 \tag{11}$$

[Raz10] that was verified in [BT11] and later in [FRV13] using the *flagmatic software* (we will discuss the latter in Section 4.1). The scale of this improvement reflects a general phenomenon: let me cautiously suggest that I am not aware of a *single* example of a *non-exact* bound in asymptotic extremal combinatorics that could not be improved by a plain application of flag algebras.

The remaining results in this section were distinctly motivated by the structure of known extremal configurations elaborated in a series of early papers by Turán [Tur41], Brown [Bro83], Kostochka [Kos82] and Fon-der-Flaass [FdF88], and we review it first. Our description (borrowed from [Raz11b]) has a rather distinct analytical flavor; for more combinatorial treatment see e.g. [Kee11, Section 7].

Let Γ be a (possibly infinite) orgraph without induced copies of $\vec{C}_4$. Let $FDF(\Gamma)$ be the 3-graph on $V(\Gamma)$ in which (u, v, w) spans an edge if and only if $\Gamma|_{\{u,v,w\}}$ contains either an isolated vertex (i.e., a vertex of both in-degree and out-degree 0) or contains a vertex of out-degree 2. Then $FDF(\Gamma)$ does not contain induced copies of I_4^3 [FdF88].

Next, let $\Omega \stackrel{\text{def}}{=} \mathbb{Z}_3 \times \mathbb{R}$, and consider the (infinite) orgraph $\Gamma_K = (\Omega, E_K)$ given by

$$E_K \stackrel{\text{def}}{=} \{ \langle (a, x), (b, y) \rangle \mid (x + y < 0 \land b = a + 1) \lor (x + y > 0 \land b = a - 1) \} .$$

Γ_K does not have induced copies of $\vec{C}_4$ and hence $FDF(\Gamma)$ does not contain induced copies of I_4^3. The set of known extremal configurations when the number of vertices is divisible by three is precisely the set of all induced subgraphs of this 3-graph that are of the form $FDF(\Gamma_K|_{\mathbb{Z}_3 \times S})$, where S is an arbitrary finite set of reals.

Turán's original configuration [Tur41] corresponds to the case when $S \subseteq \mathbb{R}^+$. Brown's examples are obtained when negative entries in S are allowed, but are always smaller in absolute values than positive entries. Kostochka's examples [Kos82] correspond to arbitrary finite S. And if we replace [the uniform measure on] S by a non-atomic measure on the real line, we will get a full description of all known $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ with $\phi(I_4^3) = 0$ and $\phi(e) = 4/9$.

Turán's original example does not contain induced copies of G_3 which implies $\pi_{\min}(I_4^3, G_3) \leq \frac{4}{9}$. Razborov [Raz10] proved that in fact

$$\pi_{\min}(I_4^3, G_3) = \frac{4}{9} \tag{12}$$

which also was the first application of the method in its genuinely plain form. Baber and Talbot [BT12, Theorem 25] gave a list of ten 3-graphs

$\{H_1, \ldots, H_{10}\}$ on six vertices for which non-induced results of the same nature hold: $\pi(K_4^3, H_i) = \frac{5}{9}$ $(1 \leq i \leq 10)$; their proof method is also plain.

Pikhurko [Pik11] proved that for a sufficiently large n, Turán's example is the *only* 3-graph on which $\mathrm{ex}_{\min}(n; I_4^3, G_3)$ is attained. This was also one of the first papers to demonstrate the three-step program for converting asymptotic flag-algebraic results into exact ones:

1. Describe the set of all extremal elements in $\mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ (which in this particular case consists of a single element).

2. Prove stability, that is that the convergence in the pointwise topology described in Section 1 can be strengthened to convergence in the edit distance.

3. Move from stability to exact results using combinatorial techniques.

Let us now take a brief de-tour and discuss a couple results of similar nature inspired by the next case $r = 3$, $\ell = 5$ in Turán's problem. The situation with computing $\pi_{\min}(I_5^3)$ itself is very similar to $\pi_{\min}(I_4^3)$: Turán's conjecture says that $\pi_{\min}(I_5^3) = 1/4$, and there are many non-isomorphic configurations realizing this bound. The simplest of them given by Turán himself is the disjoint union $K_{n/2}^3 \dot\cup K_{n/2}^3$ of two cliques of the same size. Let H_1, H_2 be the two non-isomorphic 3-graphs on 5 vertices with precisely two edges. Then they are missing in Turán's example above, and Falgas-Ravry and Vaughan proved in [FRV13] that

$$\pi_{\min}(I_5^3, H_1, H_2) = 1/4$$

which is analogous to (12). Their proof method is plain.

To review another remarkable result, it is convenient to switch to the dual notation. Turán's construction from the previous paragraph implies that $\pi(K_5^3) \geq 3/4$ and, more generally, $\pi(G) \geq 3/4$ for any 3-graph G that is not 2-colorable. In particular, this applies to critical (that is, edge minimal) 3-graphs on six vertices with chromatic number 3. There are precisely six such graphs; one of them being K_5^3 plus an isolated vertex (in other words, $\bar{J}_5$) and, obviously, $\pi(K_5^3) = \pi(\bar{J}_5)$. Quite remarkably, using flag algebras, Baber [Bab11, Theorem 2.4.1] proved that $\pi(G) = 3/4$ for every one of the remaining five graphs on the list; his proof is plain.

We now return to the tetrahedron problem. Clearly, not all graphs without copies of I_4^3 can be realized in the form $FDF(\Gamma)$, and Fon-der-Flaass

[FdF88] asked whether Turán's conjecture can at least be proved for 3-graphs of his special form. He himself showed a lower bound of 3/7 (superseded by (11)). While the Fon-der-Flaass conjecture is still open, Razborov [Raz11b] verified it under either one of the two following assumptions:

1. Γ is an orientation of a complete t-partite graph (not necessarily balanced) for some t;

2. The edge density of Γ is $\geq 2/3 - \epsilon$ for some absolute constant ϵ.

Note that 2) settles a local version of the Fon-der-Flaass conjecture, that is proves it in an open neighborhood of the set $\Phi \subseteq \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ of known extremal configurations. The proof method is a combination of plain and more sophisticated techniques heavily based upon working with several different kinds of combinatorial structures at once and frequently transferring auxiliary results from one context to another. The author expresses his hope that this kind of interaction (mostly human reasoning aided in appropriate places by the hammer-like power of plain flag-algebraic arguments) will become increasingly more popular in the area.

The result (12) is relevant only to the original extremal example given by Turán as all others contain plenty of induced copies of G_3. Razborov [Raz12] identified three 3-graphs on 5 vertices given by their set of edges as follows:

$$\begin{aligned} E(H_1) &\stackrel{\text{def}}{=} \{(123)(124)(134)(234)(125)(345)\} \\ E(H_2) &\stackrel{\text{def}}{=} \{(123)(124)(134)(234)(135)(145)(235)(245)\} \\ E(H_3) &\stackrel{\text{def}}{=} \{(123)(124)(134)(234)(125)(135)(145)(235)(245)\} \end{aligned}$$

and proved that

$$\pi_{\min}(I_4^3, H_1, H_2, H_3) = 4/9. \tag{13}$$

The motivation behind this result is that, as induced subgraphs, H_1, H_2, H_3 are missing in $FDF(\Gamma_K)$ and, thus, in all known extremal configurations. Flag algebras are used in this proof only "behind the scene", but the proof method itself deserves a few words here. Let us call a 3-graph H *singular* if its edge set is *not* a superset of $E(FDF(\Gamma))$ for any orgraph Γ which is an orientation of a complete t-partite graph (cf. the first result from [Raz11b] cited in item 1) on page 16) and does not contain induced copies of $\vec{C}_4$. Then [Raz12] proved that

$$\widehat{\pi}_H(H_1, H_2, H_3) = 0,$$

where H is an *arbitrary* singular 3-graph and $\widehat{\pi}_H(\mathcal{F})$ is defined similarly to $\pi_H(\mathcal{F})$, with the difference that only induced copies of elements in $\mathcal{F}$ are forbidden. The proof uses Ramsey theory, and the main result (13) follows almost immediately from this and the first result in [Raz11b]. We are not aware of a similar "zero-inducibility" phenomenon that would not have held for some trivial reasons.

In conclusion, [Raz11b, Raz12] provide several results that verify Turán's conjecture for several natural classes containing the set Φ_{Turan} of all known extremal examples. None of them, however, covers an *open* neighborhood of Φ_{Turan}, and we believe that obtaining such a local result would have been a major step toward resolving the unrestricted version of the tetrahedron problem.

3.3. Caccetta-Häggkvist conjecture

In this section we work with oriented graphs.

In 1970, Behzad, Chartrand and Wall [BCW70] asked the following question: if G is a bi-regular orgraph on n vertices of girth $(\ell + 1)$ (i.e., $\vec{C}_k$-free for any $k \leq \ell$), how large can be its degree? They conjectured that the answer is $\lfloor \frac{n-1}{\ell} \rfloor$ and presented a simple construction attaining this bound. Eight years later, Caccetta and Häggkvist [CH78] proposed to lift in this conjecture the restriction of bi-regularity and, moreover, restrict attention to minimal *outdegree* only. In other words, they asked if every orgraph without oriented cycles of length $\leq \ell$ must contain a vertex of out-degree $\leq \frac{n-1}{\ell}$, and it is this question that became known as the *Caccetta-Häggkvist conjecture*. Like in the previous section 3.2 we concentrate on the case $\ell = 3$ even if some prominent work has been done for higher values of ℓ.

Let c be the minimal x for which every $\vec{C}_3$-free orgraph on n vertices contains a vertex of outdegree $\leq (c + o(1))n$; the Caccetta-Häggkvist conjecture then says[5] that $c = 1/3$. Caccetta and Häggkvist themselves proved the bound $c \leq \frac{3-\sqrt{5}}{2} \approx 0.382$ [CH78]. In the paper [Bon97] that, as we acknowledged in Section 1, was one of the predecessors of flag algebras, Bondy proved that $c \leq \frac{2\sqrt{6}-3}{5} \leq 0.379$. His proof is essentially what we would call here a plain application of the method using $\vec{C}_3$-free orgraphs on 4 vertices (there are 32 of them). However, instead of actually solving the resulting SDP, Bondy gives a hand-manufactured (non-optimal) solution to it. Shen

[5]It is well-known that its asymptotic and exact versions are equivalent.

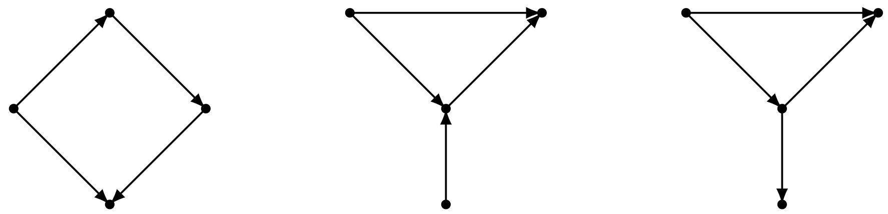


Figure 1: Forbidden orgraphs.

[She98] improved this to $c \leq 0.3543$, and Hamburger, Haxell and Kostochka [HHK07] proved a bound of $c \leq 0.3532$.

The current record of $c \leq 0.3465$ was established by Hladký, Král' and Norin in [HKN09] using flag algebras. After incorporating an inductive argument previously used by Shen in [She98], their proof method is mostly plain, but it also introduces one more novel and important feature. Namely, [HKN09] utilizes a result by Chudnovsky, Seymour and Sullivan [CSS08] on eliminating cycles in triangle-free digraphs that is only *somewhat* related to the Caccetta-Häggkvist conjecture, and adding that auxiliary result to the computational brew is paramount for the improvement. Again, I would like to express my hope that in future we will see more examples of interaction of this sort between different problems.

As for partial but exact results, Razborov [Raz11a] proved the Caccetta-Häggkvist conjecture under the additional assumption that the three orgraphs on Figure 1 are missing as induced subgraphs. Like in the previous section, the point here is that these orgraphs are missing in all known extremal configurations; for the description of the latter see [Bon97, Section 3] and [Raz11a, Section 2]. The proof is not plain and in fact does not use Cauchy-Schwarz at all. Moreover, all concrete calculations are so simple that the proof was presented in an entirely finite setting but using flag-algebraic notation.

3.4. Topics in hypergraphs motivated by the Erdős-Stone-Simonovits theorem

From this point on, all flag-algebraic proofs we review are plain. Therefore, we will normally omit this qualification.

As we already noted in Section 2, the Erdős-Stone-Simonovits theorem (3) completely settles the question of computing $\pi(\mathcal{F})$ for finite families of *ordinary* graphs. But it also implies several interesting *structural* consequences pertaining to the behavior of this function. In this section we survey a few contributions to the hypergraph theory bound together by the general intention to understand how precisely badly does this theorem fail for hypergraphs.

To start with, (3) implies that $\pi(\mathcal{F}) = \min_{F\in\mathcal{F}} \pi(F)$ (one direction is obvious), and by analogy with objects like principal ideals etc. it is natural to say that for ordinary graphs the function $\pi(\mathcal{F})$ displays *principle behavior.* It is also natural to ask if this is true for hypergraphs, and, indeed, Mubayi and Rödl [MR02] conjectured that *non-principal families* $\mathcal{F}$ (i.e., those for which $\pi(\mathcal{F}) < \min_{F\in\mathcal{F}} \pi(F)$) exist already for 3-graphs. They further conjectured that they exist even with $|\mathcal{F}| = 2$.

The first question was answered in affirmative by Balogh [Bal02], but, in his own words, "the cardinality of the set $\mathcal{F}$ is not immediately obvious". The second question was answered by Mubayi and Pikhurko [MP08] who showed that the pair (K_4^3, J_5) is not principal.

From the discussion in [MR02] it is sort of clear that the authors expect the pair $(G_3, \mathcal{C}_5)$ to be non-principal, and to that end they note the known inequality

$$\pi(G_3) \geq \frac{2}{7} \text{ [FF84]}, \tag{14}$$

as well as prove new results $\pi(\mathcal{C}_5) \geq 0.464$ and $\pi(G_3, \mathcal{C}_5) \leq \frac{10}{31}$.

Using flag algebras, Razborov [Raz10] improved the latter bound to

$$\pi(G_3, \mathcal{C}_5) \leq 0.2546 < \frac{2}{7}$$

thus proving that $(G_3, \mathcal{C}_5)$ is indeed a non-principal pair. Then Falgas-Ravry and Vaughan [FRV13], also using flag algebras, proved that the pairs $(G_3, F_{3,2})$ and (K_4^3, J_4) are non-principal. The former example is remarkable since they were also able to compute

$$\pi(G_3, F_{3,2}) = \frac{5}{18},$$

and $\pi(F_{3,2}) = \frac{4}{9}$ had been known before [FPS03] (for a several-line flag-algebraic proof of this result see [Raz10, Theorem 5]). Nonetheless, $\pi(G_3)$ is still unknown, and it is interesting to note in this respect that we still do not know of any example of a non-principal family $\mathcal{F}$ for which we can *actually* compute all involved quantities $\pi(\mathcal{F})$ and $\pi(F)$ $(F \in \mathcal{F})$.

Another obvious consequence of the Erdős-Stone-Simonovits theorem is that for ordinary graphs, $\pi(\mathcal{F})$ is always rational. The book [CG98] mentions the conjecture, believed to be due to Erdős, that this will also be the case for r-graphs. This conjecture was disproved using flag algebras by Baber and Talbot [BT12] who gave a family of three 3-graphs $\mathcal{F}$ such that $\pi(\mathcal{F}) = \lambda(F_{3,2}) = \frac{189+15\sqrt{15}}{961}$. It was also independently disproved by Pikhurko using different methods [Pik12b], but his family $\mathcal{F}$ is huge.

Yet another consequence of the Erdős-Stone-Simonovits theorem is that in case of ordinary graphs, for any $\alpha \in [0, 1)$ the density bound $\pi(\mathcal{F}) \leq \alpha$ can be forced by a finite family $\mathcal{F}$ such that all graphs $G \in \mathcal{F}$ have larger density $\geq \beta$ for some fixed $\beta > \alpha$. Moreover, the graphs $G \in \mathcal{F}$ can be made arbitrarily large, and (this is important!) β does not depend on $\min_{G\in\mathcal{F}} |V(G)|$. For example, if $\alpha \in [1/2, 2/3)$, then this property is witnessed by taking $\beta = 2/3$ and letting $\mathcal{F}$ consist of a single balanced complete tri-partite graph. α is said to be a *jump* for an integer $r \geq 2$ if the analogous property holds for r-graphs.

Erdős [Erd71] showed that for all r, every $\alpha \in \left[0, \frac{r!}{r^r}\right)$ is a jump and conjectured that, like in the case of ordinary graphs, every $\alpha \in [0, 1)$ is a jump. And perhaps the most surprising fact about jumps is that this "jumping constant conjecture" is not true. The first examples of non-jumps were given by Frankl and Rödl in [FR84], and a number of other examples followed. All of them, however, live in the interval $\left[\frac{5r!}{2r^r}, 1\right)$, and what happens in between (i.e., for $\alpha \in \left[\frac{r!}{r^r}, \frac{5r!}{2r^r}\right)$) was a totally grey area.

Using flag algebras, Baber and Talbot [BT11] gave the first example of jumps in this intermediate interval by showing that all $\alpha \in [0.2299, 0.2316)$ are jumps for $r = 3$. Their proof also uses a previous characterization from [FR84] that allows to get rid of the condition that $G \in \mathcal{F}$ must be arbitrarily large by considering their Lagrangians instead. Given this reduction, Baber and Talbot produced a set $\mathcal{F}$ consisting of five 3-graphs on 6 vertices such that $\lambda(F) \geq 0.2316$ $(F \in \mathcal{F})$ while $\pi(\mathcal{F}) \leq 0.2299$. It is worth noting that whether $\alpha = 2/9$ is a jump for $r = 3$ (which was one of the questions asked in the original paper by Erdős) still remains open.

3.5. Induced H-densities

So far we predominantly dealt with "normal" Turán densities $\pi(\mathcal{F})$, $\pi_{\min}(\mathcal{F})$, i.e. special cases of $\pi_H(\mathcal{F})$, $\pi_{\min,H}(\mathcal{F})$ when H is a (hyper)edge. In this section on the contrary we review a few results proven with the help of flag algebras in which the graph H is more complicated.

Triangle-free graphs need not be bipartite. But how exactly far from being bipartite can they be? In 1984, Erdős [Erd84, Questions 1 and 2] considered three quantitative refinements of this question, and one of them was to determine $\mathrm{ex}_{C_5}(n, K_3)$. Györi [Gyo89] had a partial result in that direction.

The asymptotic version of Erdős's question was solved using flag algebras by H. Hatami et. all [HHK+11] and Grzesik [Grz12] who independently proved that

$$\pi_{C_5}(K_3) = \frac{5!}{5^5}. \tag{15}$$

The standard trick with blow-ups (cf. (10)) immediately implies that

$$\mathrm{ex}_{C_5}(5\ell, K_3) = \ell^5. \tag{16}$$

[HHK+11] also proved that the infinite blow-up of C_5 is the only element $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ realizing equality in (15), and that the finite balanced blow-up of C_5 is the only graph realizing equality in (16). Remarkably, the proof of the latter result *bypasses the stability approach* outlined in Section 3.2. Namely, given a finite K_3-free graph G, instead of viewing this graph as a member of a converging sequence, [HHK+11] simply considers its infinite blow-up $\phi_G \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ and *directly* applies to it the asymptotic uniqueness result.

When $n = 5\ell + a$ $(1 \leq a \leq 4)$, [HHK+11] also proved that $\mathrm{ex}_{C_5}(n, K_3) = \ell^{5-a}(\ell + 1)^a$, the equality being attained at almost balanced blow-ups of C_5. But this proof already uses the traditional stability approach, and, as a consequence, works only for sufficiently large n.

Somewhat similar in spirit is another question asked by Erdős in [Erd62]. The Ramsey theorem is equivalent to the statement that for any fixed $k, \ell > 0$, for all sufficiently large n we have $\mathrm{ex}_{\min,I_k}(n, K_\ell) > 0$. Erdős asked about the *quantitative* behavior of this function and conjectured that the minimum is attained for the balanced $(\ell - 1)$-partite graph. Asymptotically, if we let

$$c_{k,\ell} \stackrel{\mathrm{def}}{=} \pi_{\min,I_k}(K_\ell),$$

Erdős's conjecture says that

$$c_{k,\ell} = (\ell - 1)^{1-k}. \tag{17}$$

This was disproved by Nikiforov [Nik01] who observed that the blow-up of C_5 we just discussed in a different context actually implies that $c_{4,3} \leq \frac{3}{25}$. Moreover, Nikiforov showed that Erdős's conjecture (17) can be true only for finitely many pairs (k, ℓ).

Using flag algebras, Das et. all [DHM+12] and Pikhurko [Pik12a] independently proved that $c_{3,4} = 1/9$ (thus confirming Erdős's conjecture in this case) and that $c_{4,3} = 3/25$ (thus showing that Nikiforov's counterexample is the worst possible). Both papers use the stability approach to get exact results for sufficiently large n. [DHM+12, Section 6] states that their unverified calculations confirm Erdős's conjecture in two more cases: $c_{3,5} = 1/16$ and $c_{3,6} = 1/25$. Both these calculations were verified by Vaughan (referred to in [Pik12a]) who also confirmed Erdős's conjecture in one more case: $c_{3,7} = 1/36$. Along the other axis, Pikhurko [Pik12a] calculated $c_{5,3}, c_{6,3}$ and $c_{7,3}$.

Let us now discuss "pure" *inducibility* $i(H)$ of a graph/orgraph/hypergraph H that in our notation is simply equal to $\pi_H(\emptyset)$.

There is one self-complimentary graph on 4 vertices, P_4 and five complimentary pairs which (since $\pi_H(\emptyset) = \pi_{\bar{H}}(\emptyset)$) give rise to six different problems of determining $\pi_H(\emptyset)$. One of them (K_4/I_4) is trivial, and two problems had been solved before with other methods.

Using flag algebras, Hirst [Hir11] solved two more cases: he showed that

$$\pi_{K_4-K_2}(\emptyset) = \frac{72}{125}$$

and that

$$\pi_{K_4-P_3}(\emptyset) = \frac{3}{8}.$$

Thus, now P_4 is the only remaining graph on 4 vertices whose inducibility is still unknown.

Sperfeld [Spe11] studied inducibility for *oriented* graphs. Using flag algebras, he showed that $\pi_{\vec{C}_3}(\emptyset) = \frac{1}{4}$ and obtained a few non-exact results improving on previous bounds: $\pi_{\vec{P}_3}(\emptyset) \leq 0.4446$ (the conjectured value is $2/5$), $\pi_{\vec{C}_4}(\emptyset) \leq 0.1104$ and $\pi_{\vec{K}_{1,2}}(\emptyset) \leq 0.4644$. Then Falgas-Ravry and Vaughan [FRV12] were able to actually compute the latter quantity:

$$\pi_{\vec{K}_{1,2}}(\emptyset) = 2\sqrt{3} - 3.$$

They also computed $\pi_{\vec{K}_{1,3}}(\emptyset)$ that turned out to be a rational function in a root of a cubic polynomial.

In the department of 3-graphs, the same paper [FRV12] calculated the inducibility of G_2:

$$\pi_{G_2}(\emptyset) = \frac{3}{4}.$$

Slightly stretching our notation, let $\pi_{m.k}(\emptyset)$ be the minimal induced density of the *collection* of all 3-graphs on m vertices with exactly k edges (thus e.g. $\pi_{G_2}(\emptyset) = \pi_{4.2}(\emptyset)$). Falgas-Ravry and Vaughan also proved in [FRV12] that

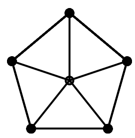
$$\pi_{5.1}(\emptyset) = \pi_{5.9}(\emptyset) = \frac{5}{8}$$

and

$$\pi_{5.k}(\emptyset) = \frac{20}{27} \ (3 \le k \le 7).$$

3.6. Miscellaneous results

A graph H is *common* if the sum of the number of its copies (not necessarily induced) in a graph G and the number of such copies in the complement of G is asymptotically minimized by taking G to be a random graph. Erdős [Erd62] conjectured that all complete graphs are common, and this conjecture was disproved by Thomason [Tho89] who showed that for $p \geq 4$, K_p is not common. It is now known that common graphs are very rare, and several authors specifically asked if the wheel W_5 shown on Figure 2 is common.



W_5

Figure 2: The 5-wheel.

This question becomes amenable to the (manifestly induced) framework of flag algebras by using the transformation (1). And, indeed, H. Hatami et. all [HHK$^+$12] proved that W_5 is common. This is the only result in our survey where optimization takes place over a rather complicated linear combination of "primary" induced densities rather than individual densities.

Erdős et. all asked in [EFGS89] a question that later became known as the *(2/3)-conjecture*. Given a 3-coloring of the edges of K_n, what is the smallest t such that there exists a color c and a set A of t vertices whose c-neighborhood has density at least 2/3? The conjecture says that $t = 3$, the previously known bound was $t = 22$, and using flag algebras, Král' et. all proved in [KLS$^+$13] that one can actually take $t = 4$.

In [KMS12], Král', Mach and Sereni looked at the following geometric problem resulted from the work by Boros and Füredi [BF84] and Bárány [Bar82]. What is the minimal constant c_d such that for every set P of n points in $\mathbb{R}^d$ in general position there exists a point of $\mathbb{R}^d$ contained in at least $c_d\binom{n}{d+1}$ d-simplices with vertices at the points of P. As stated, it is not amenable to the approach of flag algebras, but Gromov [Gro10] was able to find a topological approach to it, and its later expositions (see [KMS12] for details) brought it rather close to that realm. One remaining concepts that still can not be handled by flag algebras in full generality is that of *Seidel minimality* as it quantifies over arbitrary sets of vertices. Král', Mach and Sereni, however, showed that by applying this property only to certain sets definable in this language they can improve known bounds on c_d.

In the rest of this section we review a few more results about 3-graphs obtained with the method of flag algebras that were not addressed in our previous account. This activity started with the *Mubayi challenge* when all exact results presented to the author by Dhruv Mubayi found their new flag-algebraic proofs in [Raz10, Section 6.2] of varying and surprisingly unpredictable computational difficulty. [Raz10] also gave a few non-exact results, of which we would like to mention here only $\pi(G_3) \leq 0.2978$ later improved by Baber and Talbot [BT11] to $\pi(G_3) \leq 0.2871$ which is already quite close to the conjectured value 2/7 (see (14)).

Baber and Talbot [BT12] go over "critical" densities 2/9, 4/9, 5/9, 3/4 (recall that $\pi(F_{3,2}) = 4/9$ and 5/9, 3/4 are conjectured values for $\pi(K_4^3), \pi(K_5^3)$, respectively). For every α from this set they were able to construct one (for $\alpha = 2/9$) or more (for $\alpha \in \{4/9, 5/9, 3/4\}$) 3-graphs F with $\pi(F) = \alpha$.

Falgas-Ravry and Vaughan [FRV13] proved, besides the results we already cited above in various contexts, several more exact results:

$$\pi(G_3, \mathcal{C}_5, F_{3,2}) = \frac{12}{49},$$

$$\pi(G_3, F_{3,2}) = \frac{5}{18},$$

$$\pi(J_4, F_{3,2}) = \frac{3}{8}.$$

In the second paper [FRV12] of the same authors they prove (again, in addition to what we already surveyed before) several more inducibility results:

$$\pi_{G_3}(K_4^3) = \frac{16}{27}, \quad \pi_{G_3}(F_{3,2}) = \frac{27}{64}, \quad \pi_{K_4^3}(F_{3,2}) = \frac{3}{32}$$
$$\pi_{G_2}(\mathcal{C}_5, F_{3,2}) = \frac{9}{16}, \quad \pi_{G_2}(G_3, F_{3,2}) = \frac{5}{9}, \quad \pi_{G_2}(G_3, \mathcal{C}_5, F_{3,2}) = \frac{4}{9}.$$

Two forthcoming papers study *codegree density* $\pi_2(F)$ for 3-graphs (see [Kee11, Section 13.2] for definitions). Falgas-Ravry, Marchant, Pikhurko and Vaughan give a new flag-algebraic proof of the result $\pi_2(F_{3,2}) = 2/3$ from Marchant's thesis. In the second paper, Falgas-Ravry, Pikhurko and Vaughan prove that $\pi_2(G_3) = 1/4$.

4. Concluding remarks

4.1. Flagmatic software

In the first few years since the inception of the method, researchers who needed it for their work had to write the code on their own, the only thing that was available from the shelf were SDP-solvers like CSDP [Bor99] or SDPA. It appears as if these home-made pieces of software greatly differ in the level of their public availability, user-friendliness and, most important, versatility. Like in many other similar scenarios, it can be expected that this period of anarchy will eventually be over, and the separation between users and developers (with the clear understanding that these two groups are likely to overlap significantly) will be defined more clearly. This will also likely imply that the many ad hoc programs around will give way to one or a few "standard" packages, and researchers that are new to the method will largely lose the initiative to program on their own.

One very serious bid to become such a "golden standard" has been made by the *Flagmatic software* developed by Vaughan and, in fact, many results that we surveyed above were obtained using this program. It is publicly available from `http://www.flagmatic.org`, and (from everything I know) it is user-friendly. Versatility is also improving: while this project started with 3-graphs, the last version 2 also has support for ordinary graphs and oriented graphs. Time will tell if Flagmatic gets a serious competitor, but at the moment this seems to be the only option for a researcher who needs to use the method on a reasonably recurrent basis but does not want to invest time into writing his/her own code.

4.2. Beyond Turán densities?

Turán densities for dense graphs is by far not the only area in discrete mathematics and beyond where Cauchy-Schwarz and positive semi-definite programming are used extensively. Thus, it is natural to wonder if formal methods similar to flag algebras can be applied elsewhere. In cases we potentially have in mind it is more or less clear how to come up with a mathematically beautiful calculus that "works in theory". But our question is more pragmatic: can it be done in such a way that it will *actually* allow to prove *new concrete* results in the area in question. See [FRV13, Section 4.1] for a very relevant discussion of the complexity barrier that (as we believe) prevents us from getting many more, and possibly very great, results with this method even on its home field, asymptotic extremal combinatorics.

We are aware of two moderate but concrete and successful steps in that direction. Baber [Bab11, Chapter 2.5] (some of these results were later independently re-discovered by Balogh et. all in [BHLL12]) extends the method to Turán densities for subgraphs of the hypercube Q_n. The latter is a rather sparse graph, so significant modifications are necessary. And Norin and Zwols (personal communication) started considering applications of the flag algebra framework to the study of crossing numbers, particularly of the complete bipartite graph $K_{n,n}$. They already were able to get a numerical improvement on the previously best known bound from [dKPS07].

One more paper that might be mentioned here is the work by Král', Mach and Sereni [KMS12] on the Boros-Füredi-Bárány problem that we already discussed in Section 3.6. But their approach is sort of opposite: they "massage" the problem they are interested in until it fits nicely the framework of flag algebras as originally defined in [Raz07].

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