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On the minimal density of triangles in graphs

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For a fixed $\rho \in [0,1]$, what is (asymptotically) the minimal possible density $g_3(\rho)$ of triangles in a graph with edge density ρ? We completely solve this problem by proving that

$$g_3(\rho) = \frac{(t-1)\left(t-2\sqrt{t(t-\rho(t+1))}\right)\left(t+\sqrt{t(t-\rho(t+1))}\right)^2}{t^2(t+1)^2},$$

where $t \stackrel{\text{def}}{=} \lfloor 1/(1-\rho)\rfloor$ is the integer such that $\rho \in \left[1-\frac{1}{t}, 1-\frac{1}{t+1}\right]$.

1. Introduction

The most famous result of extremal combinatorics is probably the celebrated theorem of Turán [20] determining the maximal number $\mathrm{ex}(n;K_r)$ of edges in a K_r-free graph on n vertices[1]. Asymptotically, $\mathrm{ex}(n;K_r) \approx \left(1-\frac{1}{r-1}\right)\binom{n}{2}$. The non-trivial part (that is, the upper bound) of this theorem in the contrapositive form can be stated as follows: any graph G with $m > \mathrm{ex}(n;K_r)$ edges contains at least one copy of K_r. The quantitative version of this latter statement (that is, how *many* such copies, as a function $f_r(m,n)$ of r,n,m, must necessarily exist in any graph G) received quite a fair attention in combinatorial literature (in more general context of arbitrary forbidden subgraphs, questions of this sort were named in [7] "the theory of supersaturated graphs") and turned out to be notoriously difficult. Erdös [4, 5] computed $f_r(m,n)$ exactly when m is very close to $\mathrm{ex}(n;K_r)$; more specifically, when

$$m \le \mathrm{ex}(n;K_r) + C_r n \tag{1.1}$$

for some constant $C_r > 0$. He also completely described extremal graphs for this case.

In this paper we are more interested in the case when m is much larger than $\mathrm{ex}(n;K_r)$ (typically, $m = \rho\binom{n}{2}$, where the edge density ρ should be thought of as a fixed constant

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[1] The case $r = 3$ had been previously solved by Mantel [14].

strictly greater than $1-\frac{1}{r-1}$), and we are interested in the asymptotic behavior of $f_r(m,n)$ as a function of ρ only:

$$f_r\left(\rho\binom{n}{2},n\right)\approx g_r(\rho)\binom{n}{r}.$$

The main result from [7] then readily implies that $g_r(\rho)>0$ as long as $\rho>1-\frac{1}{r-1}$.

It turns out that the values $\rho=1-\frac{1}{t}$ (where $t\geq r-1$ an integer), that is those for which there exists an almost balanced complete t-partite graph, are critical and play an extremely critical role in the whole analysis. To start with, the analytical expression of the best known upper bound on $g_r(\rho)$ is different in every interval $\rho\in\left[1-\frac{1}{t},1-\frac{1}{t+1}\right]$. It corresponds to complete $(t+1)$-partite graphs in which t parts are roughly equal and larger than the remaining part, and has the form

$$\left.\begin{aligned}&g_r(\rho)\leq\frac{(t-1)!}{(t-r+1)!(t(t+1))^{r-1}}\cdot\left(t-(r-1)\sqrt{t(t-\rho(t+1))}\right)\\&\cdot\left(t+\sqrt{t(t-\rho(t+1))}\right)^{r-1}\quad\left(\rho\in\left[1-\frac{1}{t},1-\frac{1}{t+1}\right]\right).\end{aligned}\right\}\tag{1.2}$$

The following conjecture is widely believed.

Conjecture 1. *The bound* (1.2) *is tight.*

As for lower bounds on $g_r(\rho)$, Goodman [11] (see also [17]) proved that

$$g_3(\rho)\geq\rho(2\rho-1),\tag{1.3}$$

and Lovász and Simonovits [13] (referring to an earlier paper by Moon and Moser [15]; see also [12]) extended this bound to arbitrary r as

$$g_r(\rho)\geq\prod_{i=1}^{r-1}(1-i(1-\rho)).\tag{1.4}$$

It can be easily seen that the lower bound (1.4) matches the upper bound (1.2) for the critical values $\rho=1-\frac{1}{t}$. Also, (1.2) is concave in every intermediate interval $\left[1-\frac{1}{t},1-\frac{1}{t+1}\right]$, whereas (1.4) is convex. In the beautiful paper [1] (for a complete proof see e.g. [2, Chapter VI.1]) Bollobás proved that in fact the piecewise linear function, linear in every interval $\left[1-\frac{1}{t},1-\frac{1}{t+1}\right]$ and coinciding with (1.2), (1.4) at its end points also provides a lower bound on $g_r(\rho)$:

$$\begin{aligned}g_r(\rho)\geq\frac{t!}{(t-r+1)!}&\left\{\left(\frac{t}{(t+1)^{r-2}}-\frac{(t+1)(t-r+1)}{t^{r-1}}\right)\rho\right.\\&\left.+\left(\frac{t-r+1}{t^{r-2}}-\frac{t-1}{(t+1)^{r-2}}\right)\right\}.\end{aligned}$$

Lovász and Simonovits [13] proved Conjecture 1 in some sub-intervals of the form $\left[1-\frac{1}{t},1-\frac{1}{t}+\epsilon_{r,t}\right]$, but (in their own words) the constant $\epsilon_{r,t}$ is so small that they even did not dare to estimate $\epsilon_{3,2}$. On the positive side, their proof sufficiently narrows down the search space for extremal graphs to compute the right value of the constant C_r in (1.1) for which Erdös's description of extremal graphs still holds: $C_r \geq \frac{1}{r-1}$, and this is best possible.

Finally, Fisher [9] proved[2] Conjecture 1 for $r = 3,\ t = 2$. His result was independently re-discovered by Razborov [19] with the help of a totally different method.

In this paper we enhance this latter proof with one more inductive argument and establish Conjecture 1 for $r = 3$ and *arbitrary* values of t. Like the "base" proof, this extension was worked out entirely in the framework of *flag algebras* developed in [19], and it is presented here within that framework. Our arguments in favour of this approach were carefully laid out in [19].

2. Notation and preliminaries

All graphs considered in this paper are undirected, without loops and multiple edges. $V(G)$ is the vertex set of a graph G, and $E(G)$ is its edge set. K_n, P_n are an n-vertex clique and an n-vertex path (of length $n-1$), respectively. $\bar{G}$ is the complement of G (on the same vertex set). All subgraphs are by default induced, and for $V \subseteq V(G)$, $G|_V$ is the subgraph induced on V and $G - V \stackrel{\text{def}}{=} G|_{V(G)\setminus V}$ is the result of removing the vertices V from the graph. A *graph embedding* $\alpha : H \longrightarrow G$ is an *injective* mapping $\alpha : V(H) \longrightarrow V(G)$ such that $(v, w) \in E(H)$ iff $(\alpha(v), \alpha(w)) \in E(G)$.

$[k] \stackrel{\text{def}}{=} \{1, 2, \ldots, k\}$. A collection $V_1, \ldots, V_t$ of finite sets is *a sunflower with center* C iff $V_i \cap V_j = C$ for every two distinct $i, j \in [t]$. $V_1, \ldots, V_t$ are called the *petals* of the sunflower. Following [6], random objects appearing in this paper are always denoted in $\boldsymbol{math\ bold\ face}$.

2.1. Flag algebras

In this section we give a digest of those definitions and results from [19] that are necessary for our purposes. But we warn the reader that *it is not our intention to make this paper self-contained* (we actually do not see any reasonable way of achieving this goal): the main purpose is simply to indicate which parts of the general theory are really needed here. Although for intuition and illustrating examples we mostly refer to [19], many definitions and results are presented at the level of generality suitable for our proof (for example, we are exclusively working with undirected graphs rather than with arbitrary universal first-order theories as in [19]). Hopefully, this will also be helpful for better understanding of their intuitive meaning.

All notation in this section is strictly consistent with [19], and *all references by default also refer to* [19].

We let $\mathcal{M}_n$ denote the set of all graphs on n vertices up to an isomorphism. A *type* is a graph σ with $V(\sigma) = [k]$ for some non-negative integer k called the *size* of σ and denoted

[2] The proof in [9] was incomplete since it implicitly used the fact that the clique polynomial of a graph has an *unique* root of the smallest modulus. This missing statement was established only in 2000 [10].

by $|\sigma|$. In this paper we, with one exception, will be interested only in the (uniquely defined) types 0,1 of sizes 0,1, respectively, and in the type E of size 2 corresponding to an edge. For a type σ, a *σ-flag* is a pair $F = (G, \theta)$, where G is a graph and $\theta : \sigma \longrightarrow G$ is a graph embedding. If $F = (G, \theta)$ is a σ-flag and $V \subseteq V(G)$ contains $\mathrm{im}(\theta)$, then the sub-flag $(G|_V, \theta)$ will be denoted by $F|_V$. Likewise, if $V \cap \mathrm{im}(\theta) = \emptyset$, we use the notation $F - V$ for $(G - V, \theta)$. A *flag embedding* $\alpha : F \longrightarrow F'$, where $F = (G, \theta)$ and $F' = (G', \theta')$ are σ-flags, is a graph embedding $\alpha : G \longrightarrow G'$ such that $\theta' = \alpha\theta$ ("label-preserving"). F and F' are isomorphic (denoted $F \approx F'$) if there is a one-to-one flag embedding $\alpha : F \longrightarrow F'$. Let $\mathcal{F}^\sigma$ be the set of all σ-flags (up to an isomorphism), and $\mathcal{F}^\sigma_\ell \stackrel{\mathrm{def}}{=} \{\, (G, \theta) \in \mathcal{F}^\sigma \mid G \in \mathcal{M}_\ell \,\}$ be the set of all σ-flags on ℓ vertices (thus, $\mathcal{M}_\ell$ can, and often will, be identified with $\mathcal{F}^0_\ell$). $\mathcal{F}^\sigma_{|\sigma|}$ consists of the single element (σ, id) ($\mathrm{id} : \sigma \longrightarrow \sigma$ the identity embedding) denoted by 1_σ or even simply by 1 when σ is clear from the context. When $G \in \mathcal{M}_\ell$, and a type σ with $|\sigma| \leq \ell$ is embeddable in G and has the property that all σ-flags resulting from such embeddings are isomorphic, we will denote this uniquely defined σ-flag by G^σ. Additionally, the edge K_2 viewed as an element of $\mathcal{F}^0_2$ will be denoted by ρ, and the corresponding 1-flag $\rho^1 \in \mathcal{F}^1_2$ will be denoted by e.

Fix a type σ of size k, assume that integers $\ell, \ell_1, \ldots, \ell_t \geq k$ are such that

$$\ell_1 + \cdots + \ell_t - k(t-1) \leq \ell, \tag{2.1}$$

and $F = (G, \theta) \in \mathcal{F}^\sigma_\ell,\ F_1 \in \mathcal{F}^\sigma_{\ell_1}, \ldots,\ F_t \in \mathcal{F}^\sigma_{\ell_t}$ are σ-flags. We define the quantity $p(F_1, \ldots, F_t; F) \in [0,1]$ as follows. Choose in $V(G)$ uniformly at random a sunflower $(\boldsymbol{V_1}, \ldots, \boldsymbol{V_t})$ with center $\mathrm{im}(\theta)$ and petals of sizes $\ell_1, \ldots, \ell_t$, respectively (the inequality (2.1) ensures that such sunflowers do exist). We let $p(F_1, \ldots, F_t; F)$ denote the probability of the event "$\forall i \in [t](F|_{\boldsymbol{V}_i} \approx F_i)$". When $t = 1$, we use the notation $p(F_1, F)$ instead of $p(F_1; F)$.

Let $\mathbb{R}\mathcal{F}^\sigma$ be the linear space with the basis $\mathcal{F}^\sigma$, i.e. the space of all formal finite linear combinations of σ-flags with real coefficients. Let $\mathcal{K}^\sigma$ be its linear subspace generated by all elements of the form

$$\widetilde{F} - \sum_{F \in \mathcal{F}^\sigma_\ell} p(\widetilde{F}, F)F, \tag{2.2}$$

where $\widetilde{F} \in \mathcal{F}^\sigma_{\tilde{\ell}}$ and $\ell \geq \tilde{\ell}$. Let

$$\mathcal{A}^\sigma \stackrel{\mathrm{def}}{=} (\mathbb{R}\mathcal{F}^\sigma)/\mathcal{K}^\sigma.$$

For two σ-flags $F_1 \in \mathcal{F}^\sigma_{\ell_1},\ F_2 \in \mathcal{F}^\sigma_{\ell_2}$ choose arbitrarily $\ell \geq \ell_1 + \ell_2 - |\sigma|$ and define their product as the element of $\mathcal{A}^\sigma$ given by

$$F_1 \cdot F_2 \stackrel{\mathrm{def}}{=} \sum_{F \in \mathcal{F}^\sigma_\ell} p(F_1, F_2; F)F. \tag{2.3}$$

$F_1 \cdot F_2$ does not depend on the choice of ℓ (modulo $\mathcal{K}^\sigma$) and defines on $\mathcal{A}^\sigma$ the structure of a commutative associative algebra with the identity element 1_σ (Lemma 2.4)[3]. $\mathcal{A}^\sigma$

[3] On the notational side, superscripts 0 and 1 always refer to the corresponding types, and $2, 3, \ldots$ refer to powers in these algebras. We realize that this solution is not perfect, but this is the best compromise we have been able to achieve without making the notation unnecessarily clumsy.

is free, that is, isomorphic to the algebra of polynomials in countably many variables (Theorem 2.7).

Given a type σ of size k, $k' \leq k$ and an injective mapping $\eta : [k'] \longrightarrow [k]$, let $\sigma|_\eta$ be the naturally induced type of size k' (that is, $(i,j) \in E(\sigma|_\eta)$ iff $(\eta(i), \eta(j) \in E(\sigma))$). For a σ-flag $F = (G, \theta)$, the $\sigma|_\eta$-flag $F|_\eta$ is defined as $F|_\eta \stackrel{\text{def}}{=} (G, \theta\eta)$. We define the normalizing factor $q_{\sigma,\eta}(F) \in [0,1]$ as follows. Let $F = (G, \theta)$; generate an injective mapping $\boldsymbol{\theta} : [k] \longrightarrow V(G)$, uniformly at random subject to the additional restriction that it must be consistent with θ on $\mathrm{im}(\eta)$ (that is, $\boldsymbol{\theta}\eta = \theta\eta$). We let $q_{\sigma,\eta}(F)$ be the probability that $\boldsymbol{\theta}$ defines a graph embedding $\sigma \longrightarrow G$ *and* the resulting σ-flag $(G, \boldsymbol{\theta})$ is *isomorphic* to F. We let

$$[\![F]\!]_{\sigma,\eta} \stackrel{\text{def}}{=} q_{\sigma,\eta}(F) \cdot F|_\eta.$$

$[\![\cdot]\!]_{\sigma,\eta}$ gives rise to a linear operator $[\![\cdot]\!]_{\sigma,\eta} : \mathcal{A}^\sigma \longrightarrow \mathcal{A}^{\sigma|_\eta}$ (Theorem 2.5). When $k' = 0$, $[\![\cdot]\!]_{\sigma,\eta}$ is abbreviated to $[\![\cdot]\!]_\sigma$.

In the same situation, let $D \stackrel{\text{def}}{=} [k] \setminus \mathrm{im}(\theta)$ and $d \stackrel{\text{def}}{=} k - k' = |D|$. For a σ-flag F, we let

$$F{\downarrow_\eta} \stackrel{\text{def}}{=} F|_\eta - \theta(D)$$

(thus, the only difference between $F{\downarrow_\eta}$ and $F|_\eta$ is that not only we unlabel vertices in $\theta(D)$ but actually *remove* them from the flag).

For $F \in \mathcal{F}_\ell^{\sigma|_\eta}$, let

$$\pi^{\sigma,\eta}(F) \stackrel{\text{def}}{=} \sum \left\{ \widehat{F} \in \mathcal{F}^\sigma_{\ell+d} \;\middle|\; \widehat{F}\Big\downarrow_\eta = F \right\}.$$

$\pi^{\sigma,\eta}$ defines an *algebra homomorphism* $\pi^{\sigma,\eta} : \mathcal{A}^{\sigma|_\eta} \longrightarrow \mathcal{A}^\sigma$ (Theorem 2.6) that is also abbreviated to π^σ when $k' = 0$.

Since $\mathcal{A}^1$ is free, we may consider its *localization* $\mathcal{A}^1_e$ with respect to the multiplicative system $\left\{ e^\ell \mid \ell \in \mathbb{Z} \right\}$ (thus, every element of $\mathcal{A}^1_e$ can be represented in the form $e^{-\ell} f$ with $f \in \mathcal{A}^1$ and $\ell \geq 0$). For a graph $G \in \mathcal{M}_\ell$, let $G^+ \in \mathcal{M}_{\ell+1}$ be obtained by adding a new vertex v_0 connected to all vertices in $V(G)$. Let $F^+(G) \in \mathcal{F}^1_{\ell+1}$ be the 1-flag resulting from G^+ by labeling the new vertex v_0; define the element $\pi^e(G) \in \mathcal{A}^1_e$ by

$$\pi^e(G) \stackrel{\text{def}}{=} e^{-\ell} \cdot F^+(G). \tag{2.4}$$

π^e defines an algebra homomorphism $\pi^e : \mathcal{A}^0 \longrightarrow \mathcal{A}^1_e$ (Theorem 2.6).

$\mathrm{Hom}(\mathcal{A}^\sigma, \mathbb{R})$ is the set of all algebra homomorphisms from $\mathcal{A}^\sigma$ to $\mathbb{R}$, and $\mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$ consists of all those $\phi \in \mathrm{Hom}(\mathcal{A}^\sigma, \mathbb{R})$ for which $\forall F \in \mathcal{F}^\sigma(\phi(F) \geq 0)$. For every $\phi \in \mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$ we actually have $\forall F \in \mathcal{F}^\sigma(\phi(F) \in [0,1])$ (cf. Definition 5), therefore $\mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$ can be considered as a subset in $[0,1]^{\mathcal{F}^\sigma}$. We endow $[0,1]^{\mathcal{F}^\sigma}$ with product topology (aka pointwise convergent topology); the resulting space is compact and metrizable. The same is true for its closed subspace $\mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$ defined in $[0,1]^{\mathcal{F}^\sigma}$ by countably many polynomial equations.

Any continuous function $f : C \longrightarrow \mathbb{R}$ defined on a closed subset $C \subseteq \mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$ is automatically bounded and attains (at least one) global minimum. Every $f \in \mathcal{A}^\sigma$ can be viewed as a continuous function on C (given by $f(\phi) \stackrel{\text{def}}{=} \phi(f)$), and the latter remark applies to these functions as well as to their (continuous) superpositions.

We rigorously define the function g_r from the Introduction as

$$g_r(x) \stackrel{\text{def}}{=} \liminf_{n\to\infty} \min\left\{ p(K_r, G_n) \mid G_n \in \mathcal{M}_n \wedge p(\rho, G_n) \geq x \right\}.$$

$g_r(x)$ is clearly monotone in x, and (by a simple edge-adding argument), it is also continuous. Alternatively,

$$g_r(x) = \min\left\{ \phi(K_r) \,\middle|\, \phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R}) \wedge \phi(\rho) \geq x \right\}, \tag{2.5}$$

(Corollary 3.4), and we will be studying it in this form.

The partial preorder $\leq_\sigma$ on $\mathcal{A}^\sigma$ is defined as

$$f \leq_\sigma g \equiv \forall \phi \in \mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})(\phi(f) \leq_\sigma \phi(g)).$$

Felix [8] and Podolski [18] independently proved that $\leq_\sigma$ is actually a partial order (that is, homomorphisms from $\mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$ can distinguish between any two different elements of $\mathcal{A}^\sigma$). We have:

$$f \leq_\sigma g \Longrightarrow [\![f]\!]_{\sigma,\eta} \leq_{\sigma|_\eta} [\![g]\!]_{\sigma,\eta}$$

and

$$f \leq_{\sigma|_\eta} g \Longrightarrow \pi^{\sigma,\eta}(f) \leq_\sigma \pi^{\sigma,\eta}(g)$$

(Theorem 3.1).

Let $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$, and let σ be any type such that $\phi(\sigma) > 0$. An *extension* of ϕ is a probability measure $\mathbf{P}^\sigma$ on Borel subsets of $\mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$ such that for any $f \in \mathcal{A}^\sigma$ we have

$$\int_{\mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})} \phi^\sigma(f) \mathbf{P}^\sigma(d\phi^\sigma) = \frac{\phi([\![f]\!]_\sigma)}{\phi(\langle\sigma\rangle)}, \tag{2.6}$$

where $\langle\sigma\rangle \stackrel{\text{def}}{=} [\![1_\sigma]\!]_\sigma = q_{\sigma,0}(1_\sigma) \cdot \sigma \in \mathcal{A}^0$. In the situations of interest to us the normalizing factor $q_{\sigma,0}$ is equal to 1 and we have:

$$\langle\sigma\rangle = \begin{cases} 1, & \text{if } \sigma = 0 \text{ or } \sigma = 1 \\ \rho, & \text{if } \sigma = E. \end{cases}$$

Extensions always exist and are uniquely defined (Theorem 3.5). We usually visualize the measure $\mathbf{P}^\sigma$ as the random homomorphism $\boldsymbol{\phi^\sigma}$ chosen according to this measure, and for an event A, "A a.e." means $\mathbf{P}[A] = 1$.

If $\boldsymbol{\phi^1}, \boldsymbol{\phi^E}$ are extensions of the same $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ and $f \in \mathcal{A}^E$ then

$$\boldsymbol{\phi^E}(f) \geq 0 \text{ a.e.} \Longrightarrow \boldsymbol{\phi^1}([\![f]\!]_{E,1}) \geq 0 \text{ a.e.}, \tag{2.7}$$

where in the subscript of $[\![f]\!]_{E,1}$ we have abbreviated to 1 the function $\eta : \{1\} \longrightarrow \{1, 2\}$ with $\eta(1) = 1$ (Theorem 3.18). Also, for every $f \in \mathcal{A}^0$ and every type σ,

$$\boldsymbol{\phi^\sigma}(\pi^\sigma(f)) = \phi(f) \text{ a.e.} \tag{2.8}$$

(Corollary 3.19).

For a type σ, $\ell \geq |\sigma|$ and $G \in \mathcal{M}_\ell$, define $\mu_\ell^\sigma(G) \in \mathbb{R}\mathcal{F}_\ell^\sigma$ as follows:

$$\mu_\ell^\sigma(G) \stackrel{\text{def}}{=} \sum \left\{ F \in \mathcal{F}_\ell^\sigma \mid F|_0 \approx G \right\}$$

(thus, this is the sum of all σ-flags based on G). In general, μ_ℓ^σ does *not* induce a linear mapping from $\mathcal{A}^0$ to $\mathcal{A}^\ell$.

For $G \in \mathcal{M}_\ell$, let

$$\partial_1 G \stackrel{\text{def}}{=} \ell(\pi^1(G) - \mu^1(G)).$$

Let $\bar{E}$ be the other type of size 2 (corresponding to non-edge), and $Fill : \mathcal{A}^{\bar{E}} \longrightarrow \mathcal{A}^E$ be the natural isomorphism defined by adding an edge between the two distinguished vertices. Let

$$\partial_E G \stackrel{\text{def}}{=} \frac{\ell(\ell-1)}{2}(Fill(\mu_\ell^{\bar{E}}(G)) - \mu_\ell^E(G)).$$

Unlike μ_ℓ^σ, ∂_1 and ∂_E do induce linear mappings $\partial_1 : \mathcal{A}^0 \longrightarrow \mathcal{A}^1$ and $\partial_E : \mathcal{A}^0 \longrightarrow \mathcal{A}^E$, respectively (Lemmas 4.2, 4.4).

Proposition 2.1 (Theorems 4.3 and 4.5 in [19]). *Let $\vec{G} = G_1, \ldots, G_h$ be fixed graphs, $\vec{a} = (a_1, \ldots, a_h) \in \mathbb{R}^h$ and $f \in C^1(U)$, where $U \subseteq \mathbb{R}^h$ is an open neighbourhood of $\vec{a}$. We let*

$$Grad_{\vec{G},\vec{a}}(f) \stackrel{\text{def}}{=} \sum_{i=1}^{h} \left.\frac{\partial f}{\partial x_i}\right|_{\vec{x}=\vec{a}} \cdot G_i;$$

this is an element of $\mathcal{A}^0$. Assume also that we are given $\phi_0 \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ with $\phi_0(\rho) > 0$ such that $\phi_0(G_i) = a_i$ $(1 \leq i \leq h)$ and for any other $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ with $(\phi(G_1), \ldots, \phi(G_h)) \in U$ we have

$$f(\phi(G_1), \ldots, \phi(G_h)) \geq f(\vec{a}).$$

Then for the extensions $\boldsymbol{\phi_0^1}, \boldsymbol{\phi_0^E}$ of the homomorphism ϕ_0 we have

$$\boldsymbol{\phi_0^1}(\partial_1 Grad_{\vec{G},\vec{a}}(f)) = 0 \ \textit{a.e.}$$
$$\boldsymbol{\phi_0^E}(\partial_E Grad_{\vec{G},\vec{a}}(f)) \geq 0 \ \textit{a.e.}$$

In particular, for every $g \in \mathcal{A}^1$,

$$\phi_0([\![(\partial_1 Grad_{\vec{G},\vec{a}}(f))g]\!]_1) = 0 \tag{2.9}$$

and for every E-flag F,

$$\phi_0([\![(\partial_E Grad_{\vec{G},\vec{a}}(f))F]\!]_E) \geq 0 \tag{2.10}$$

(Corollary 4.6).

3. Main result

Theorem 3.1. *For every integer $t \geq 1$ and $x \in \left[1 - \frac{1}{t}, 1 - \frac{1}{t+1}\right]$,*

$$g_3(x) = \frac{(t-1)\left(t - 2\sqrt{t(t - x(t+1))}\right)\left(t + \sqrt{t(t - x(t+1))}\right)^2}{t^2(t+1)^2}. \tag{3.1}$$

Proof. The (easy) upper bound was mentioned in the Introduction. For the lower bound we proceed by induction on t. If $t = 1$ then the right-hand side of (3.1) is equal to 0 and there is nothing to prove. So, we assume that $t \geq 2$, and that the desired lower bound on $g_3(x)$ is already proved for all $s < t$ and all $x \in \left[1 - \frac{1}{s}, 1 - \frac{1}{s+1}\right]$.

Before starting the formal argument, let us give a general overview of our strategy. First of all, assuming for the sake of contradiction that there exist elements of $\mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ violating the lower bound in (3.1), we can *fix once and for all* $\phi_0 \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ that violates it most, i.e., ϕ_0 minimizes the "defect" functional $\phi(K_3) - g_3(\phi(\rho))$ over $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ with $\phi(\rho) \in \left[1 - \frac{1}{t}, 1 - \frac{1}{t+1}\right]$. Goodman's bound (1.3) implies that $\phi_0(\rho)$ is an *internal* point of this interval which allows us to conclude that ϕ_0 satisfies the two "variational principles" (corresponding to vertex and edge deletion, respectively) from Proposition 2.1.

A relatively simple calculation in flag algebras leads to the first key inequality (3.6). Intuitively, it means that *Conjecture 1 for* $r = 4$ *and any given* t *implies the same conjecture for* $r = 3$ (*and the same* t), that is exactly what we are proving. Since Conjecture 1 is trivial for $r = 4,\ t = 2$, (3.6) already implies Fisher's result (this part was actually found a year before its generalization to arbitrary t; see [19, Section 5]).

Unfortunately, despite significant effort we have not been able to prove Conjecture 1 for $r = 4$ even in the simplest non-trivial case $t = 3$. What we, however, managed to do was to show a priori weaker and more complicated lower bound on the density of K_4 (see (3.25)) that, moreover, substantially uses the extremal properties of ϕ_0. Even this lower bound, however, allows us to put things together.

Even with all numerical simplifications provided by the framework of flag algebras (where we at least do not have to worry about low-order terms!), proving (3.25), which is the key part of our argument, still involves rather tedious (but straightforward) analytical computations. The best way to see why and how it works out is to check all these calculations and constructions against the "real" extremal homomorphism corresponding to the $(t+1)$-partite graph with appropriate densities of the parts.

Besides these annoying technicalities, the proof of (3.25) involves only one new combinatorial idea, and this is where we use our inductive assumption (on t). In the formalism of flag algebras this idea is captured by the homomorphism π^e defined by (2.4), and intuitively we do the following. Fix any vertex v in our graph G and consider the subgraph G_v induced by its neighbours. Then it turns out that the constraints imposed by Proposition 2.1 imply that the edge density in G_v is at most $\left(1 - \frac{1}{t}\right)$ (3.15). Hence we can apply the inductive hypothesis to bound the density of triangles in G_v which is also the (properly normalized) density of those K_4 in G that contain v. Unfortunately, the resulting bound (3.16) is not linear in the degree of v, and converting it to the linear form (3.24) is where the most tedious analytical work occurs. After this is done, however, the required lower bound (3.25) on the density of K_4 in G is attained simply by averaging over v.

We now begin the formal proof. Denote the right-hand side of (3.1) by $h_t(x)$, and consider the continuous real-valued function f on the closed set

$$C \stackrel{\text{def}}{=} \left\{ \phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R}) \,\middle|\, \phi(\rho) \in \left[1 - \frac{1}{t}, 1 - \frac{1}{t+1}\right] \right\}$$

given by $f(\phi) \stackrel{\text{def}}{=} \phi(K_3) - h_t(\phi(\rho))$. Due to (2.5), we only have to prove that $f(\phi) \geq 0$ on C. Let $\phi_0 \in C$ be the point where f attains its global minimum. If $\phi_0(\rho) = 1 - \frac{1}{t}$ or $\phi_0(\rho) = 1 - \frac{1}{t+1}$ then we are done by Goodman's bound (1.3). Thus, we can assume that ϕ_0 is internal in C, that is, $1 - \frac{1}{t} < \phi_0(\rho) < 1 - \frac{1}{t+1}$.

The next part of our argument very closely follows [19, Section 5]. Let $a \stackrel{\text{def}}{=} \phi_0(\rho)$ and $b \stackrel{\text{def}}{=} \phi_0(K_3)$. In the setting of Proposition 2.1, let $h := 2$, $\vec{G} := \rho, K_3$, $\vec{a} = a, b$ and (with slight abuse of notation) $f(x, y) = y - h_t(x)$ be the C^1-function in the open neighbourhood $U \stackrel{\text{def}}{=} \left\{ (x, y) \,\middle|\, 1 - \frac{1}{t} < x < 1 - \frac{1}{t+1} \right\}$ of the point $(a, b) \in \mathbb{R}^2$. Then

$$\begin{aligned}
&Grad_{\rho,K_3,a,b}(f) = K_3 - h_t'(a)\rho \\
&\partial_1 Grad_{\rho,K_3,a,b}(f) = (3\pi^1(K_3) - 2h_t'(a)\pi^1(\rho)) - (3K_3^1 - 2h_t'(a)e) \\
&\partial_E Grad_{\rho,K_3,a,b}(f) = h_t'(a) \cdot 1_E - 3K_3^E.
\end{aligned}$$

Let $\boldsymbol{\phi_0^1}, \boldsymbol{\phi_0^E}$ be the extensions of ϕ_0. Applying Proposition 2.1 and using (2.8) in case of $\boldsymbol{\phi_0^1}$, we get

$$\boldsymbol{\phi_0^1}(3K_3^1 - 2h_t'(a)e) = 3b - 2ah_t'(a) \text{ a.e.} \tag{3.2}$$

and

$$\boldsymbol{\phi_0^E}(K_3^E) \leq \frac{1}{3}h_t'(a) \text{ a.e.} \tag{3.3}$$

We reserve these two facts for later use, and for the time being we only need their "light" versions (2.9) (with $g := e$) and (2.10) (with $F := \bar{P}_3^E$):

$$\phi_0(3[\![eK_3^1]\!]_1 - 2h_t'(a)[\![e^2]\!]_1) = a(3b - 2ah_t'(a)), \tag{3.4}$$

$$\phi_0([\![\bar{P}_3^E K_3^E]\!]_E) \leq \frac{1}{3}h_t'(a)\phi_0([\![\bar{P}_3^E]\!]_E) = \frac{1}{9}h_t'(a)\phi_0(\bar{P}_3). \tag{3.5}$$

Lemma 3.2. $3[\![eK_3^1]\!]_1 + 3[\![\bar{P}_3^E K_3^E]\!]_E \geq 2K_3 + K_4$.

Proof of Lemma 3.2. Both sides of this inequality can be evaluated as linear combinations of those graphs in $\mathcal{M}_4$ that contain at least one triangle. There are only four such graphs, and the lemma is easily verified by computing coefficients in front of all of them. □

Applying ϕ_0 to the inequality of Lemma 3.2, comparing the result with (3.4), (3.5) and re-grouping terms, we get

$$h_t'(a)\phi_0\left(\frac{1}{3}\bar{P}_3 + 2[\![e^2]\!]_1\right) + b(3a - 2) \geq 2h_t'(a)a^2 + \phi_0(K_4).$$

Next, $\frac{1}{3}\bar{P}_3 + 2[\![e^2]\!]_1 = K_3 + \rho$. This finally gives us

$$b(h_t'(a) + 3a - 2) \geq a(2a - 1)h_t'(a) + \phi_0(K_4).$$

The function $h_t(x)$ is concave on $\left[1-\frac{1}{t}, 1-\frac{1}{t+1}\right]$ and $h_t'\left(1-\frac{1}{t+1}\right) = \frac{3(t-1)}{t+1}$ which implies $h_t'(a) > \frac{3(t-1)}{t+1} \geq 1$ and hence $h_t'(a)+3a-2>0$. Therefore,

$$b \geq \frac{a(2a-1)h_t'(a)+\phi_0(K_4)}{h_t'(a)+3a-2}. \tag{3.6}$$

It is easy to check (see the discussion at the beginning of this section) that if we replace $\phi_0(K_4)$ in the right-hand side of (3.6) by the conjectured value of $g_4(a)$ (given in (1.2)) then this bound evaluates exactly to $h_t(a)$ and we are done. In other words, Conjecture 1 for $r=4$ and any given t implies this conjecture for $r=3$ (and the same t). In particular, since $g_4(a)=0$ for $1/2 \leq a \leq 2/3$, the case $t=2$ is already solved (this is exactly the content of [19, Section 5]), and in what follows we will assume $t \geq 3$. We can also assume w.l.o.g. that

$$b \leq h_t(a) \tag{3.7}$$

since otherwise we are already done. Let us first apply (2.7) (with $f := \frac{1}{3}h_t'(a) - K_3^E$) to (3.3). Since $[\![K_3^E]\!]_{E,1} = K_3^1$ and $[\![1_E]\!]_{E,1} = e$, we get

$$\boldsymbol{\phi}_{\mathbf{0}}^{\mathbf{1}}(K_3^1) \leq \frac{1}{3}h_t'(a)\boldsymbol{\phi}_{\mathbf{0}}^{\mathbf{1}}(e) \text{ a.e.} \tag{3.8}$$

On the other hand, (3.2) allows us to express $\boldsymbol{\phi}_{\mathbf{0}}^{\mathbf{1}}(K_3^1)$ as a linear function in $\boldsymbol{\phi}_{\mathbf{0}}^{\mathbf{1}}(e)$:

$$\boldsymbol{\phi}_{\mathbf{0}}^{\mathbf{1}}(K_3^1) = A\boldsymbol{\phi}_{\mathbf{0}}^{\mathbf{1}}(e) - B \text{ a.e.} \tag{3.9}$$

where we introduced the notation

$$\begin{aligned} A &\stackrel{\text{def}}{=} \frac{2}{3}h_t'(a), \\ B &\stackrel{\text{def}}{=} Aa - b = \frac{2}{3}ah_t'(a) - b. \end{aligned}$$

(3.7) implies

$$B \geq \frac{2}{3}ah_t'(a) - h_t(a) = \frac{(t-1)\left(t+\sqrt{t(t-a(t+1))}\right)^2}{t(t+1)^2} > 0, \tag{3.10}$$

and, thus, from (3.9) we see that $\boldsymbol{\phi}_{\mathbf{0}}^{\mathbf{1}}(\boldsymbol{e}) \geq \frac{B}{A} > 0$ a.e. On the other hand, from comparing (3.8) and (3.9), $\boldsymbol{\phi}_{\mathbf{0}}^{\mathbf{1}}(e) \leq \frac{2B}{A}$ a.e. Summarizing,

$$0 < \boldsymbol{\phi}_{\mathbf{0}}^{\mathbf{1}}(e) \leq \frac{2B}{A} \text{ a.e.} \tag{3.11}$$

Consider now any *individual* $\phi^1 \in \mathrm{Hom}^+(\mathcal{A}^1, \mathbb{R})$ for which (3.9) and (3.11) hold. Since $\phi^1(e) > 0$, ϕ^1 can be extended to a homomorphism from the localization $\mathcal{A}_e^1$ into the reals; composing it with the homomorphism π^e (given by (2.4)), we get a homomorphism $\psi \stackrel{\text{def}}{=} \phi^1\pi^e \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$. By (3.9), we have

$$\psi(\rho) = \phi^1(e^{-2}K_3^1) = \frac{\phi^1(K_3^1)}{\phi^1(e)^2} = \frac{A\phi^1(e) - B}{\phi^1(e)^2}.$$

In order to simplify this expression, we introduce new coordinates

$$\begin{aligned} z &\stackrel{\text{def}}{=} \frac{\phi^1(e)}{A}, \\ \mu &\stackrel{\text{def}}{=} \frac{B}{A^2}. \end{aligned} \tag{3.12}$$

In these coordinates,

$$\psi(\rho) = \frac{z-\mu}{z^2} \tag{3.13}$$

(which, in particular, implies $z \geq \mu$) and the second inequality in (3.11) is re-written as $z \leq 2\mu$.

Before proceeding any further, we need to localize the value of μ to the interval

$$\frac{t}{4(t-1)} \leq \mu \leq \frac{(t-1)}{4(t-2)}. \tag{3.14}$$

For the lower bound, we use the already known bound (3.10) on B followed by a simple computation:

$$\mu \geq \frac{\frac{2}{3}ah'_t(a) - h_t(a)}{\left(\frac{2}{3}h'_t(a)\right)^2} = \frac{t}{4(t-1)}.$$

For the other part, we use Goodman's basic bound $b \geq a(2a-1)$ (see (1.3)). This implies $\mu = \frac{aA-b}{A^2} \leq \frac{aA-a(2a-1)}{A^2}$. Maximizing this expression in A gives us $\mu \leq \frac{a}{4(2a-1)}$ which is at most $\frac{t-1}{4(t-2)}$ since $a \geq \frac{t-1}{t}$.

Now, as a function in z, the right-hand side of (3.13) is increasing in the interval $z \in [\mu, 2\mu]$. Therefore,

$$\psi(\rho) \leq \frac{1}{4\mu} \leq 1 - \frac{1}{t} \tag{3.15}$$

by (3.14), and we can apply the inductive hypothesis to the homomorphism ψ and conclude that

$$\psi(K_3) \geq h_s(\psi(\rho)) = h_s\left(\frac{z-\mu}{z^2}\right),$$

where $1 \leq s \leq t-1$ is chosen in such a way that

$$1 - \frac{1}{s} \leq \frac{z-\mu}{z^2} \leq 1 - \frac{1}{s+1}.$$

But $\psi(K_3) = \frac{\phi^1(K_4^1)}{\phi^1(e)^3}$, so we finally get

$$\frac{\phi^1(K_4^1)}{A^3} \geq z^3 \cdot h_s\left(\frac{z-\mu}{z^2}\right); \tag{3.16}$$

recall that ϕ^1 is an arbitrary element of $\mathrm{Hom}^+(\mathcal{A}^1, \mathbb{R})$ for which (3.9) and (3.11) hold.

The next step is to linearize this bound in z so that we can average the result according to the distribution $\boldsymbol{\phi_0^1}$, and the main technical complication is that we need our linear bound not to depend on s at all.

Recall that in the interval $z \in [\mu, 2\mu]$ the function $\frac{z-\mu}{z^2}$ is increasing from 0 to $\frac{1}{4\mu} \in \left[1 - \frac{1}{t-1}, 1 - \frac{1}{t}\right]$. For $1 \leq s \leq t-1$, denote by $\eta_s \in [\mu, 2\mu]$ the uniquely defined root of the equation

$$\frac{\eta_s - \mu}{\eta_s^2} = 1 - \frac{1}{s} \tag{3.17}$$

(so that $\eta_1 = \mu$), and let also $\eta_t \stackrel{\text{def}}{=} 2\mu$ be the right end of our interval. The right-hand side of (3.16) is continuous on $[\eta_1, \eta_t]$, and in every sub-interval $[\eta_s, \eta_{s+1}]$ $(1 \leq s \leq t-1)$ it is equal to the smooth function

$$\theta_s(z) \stackrel{\text{def}}{=} z^3 \cdot h_s\left(\frac{z-\mu}{z^2}\right).$$

The following claim will be needed only for $s = t-1$, but the general case is no harder (and is helpful for understanding the whole picture).

Claim 3.3. *For every $s = 1, 2, \ldots, t-1$, $\theta_s(z)$ is concave in the interval $z \in [\eta_s, \eta_{s+1}]$.*

Proof of Claim 3.3. First note that

$$\frac{z-\mu}{z^2} \leq \frac{\eta_{s+1} - \mu}{\eta_{s+1}^2} = 1 - \frac{1}{s+1},$$

hence $sz^2 - (s+1)(z-\mu) \geq 0$. Denoting

$$\xi \stackrel{\text{def}}{=} \sqrt{s(sz^2 - (s+1)(z-\mu))}, \tag{3.18}$$

it is easy to check that

$$\theta_s''(z) = -\frac{3(4s\mu - s - 1)^2(s-1)}{2\xi(2\xi - 2sz + s + 1)^2} \leq 0.$$

□

Now we are ready to describe our linearization of the bound (3.16). It interpolates this bound at the point η_{t-1} and has the slope $\frac{3}{2}(1-2\mu)$. That is, we claim that for all $s = 1, 2, \ldots, t-1$ and $z \in [\eta_s, \eta_{s+1}]$,

$$\left.\begin{aligned} \theta_s(z) &\geq \frac{3}{2}(1-2\mu)(z - \eta_{t-1}) + \theta_{t-1}(\eta_{t-1}) \\ &= \frac{3}{2}(1-2\mu)(z-\eta_{t-1}) + \eta_{t-1}^3 \cdot \frac{(t-2)(t-3)}{(t-1)^2}. \end{aligned}\right\} \tag{3.19}$$

We consider separately two cases, $z \geq \eta_{t-1}$ and $z \leq \eta_{t-1}$, and it is clearly sufficient to establish that

$$\theta_{t-1}'(z) \geq \frac{3}{2}(1-2\mu)\ (z \in [\eta_{t-1}, \eta_t]), \tag{3.20}$$

$$\theta_s'(z) \leq \frac{3}{2}(1-2\mu)\ (s \leq t-2,\ z \in [\eta_s, \eta_{s+1}]). \tag{3.21}$$

Proof of (3.20)**.** By Claim 3.3, we may assume w.l.o.g. that $z = \eta_t = 2\mu$, in which case

(3.20) simplifies to

$$\left.\begin{aligned}\theta'_{t-1}(2\mu) &= \frac{3(t-2)}{(t-1)t^2}\left((t-1)\mu - \sqrt{(t-1)\mu(4(t-1)\mu-t)}\right)\\ &\times\left(8(t-1)\mu - t + 4\sqrt{(t-1)\mu(4(t-1)\mu-t)}\right) \geq \frac{3}{2}(1-2\mu).\end{aligned}\right\} \quad (3.22)$$

Recall from (3.14) that $\mu \in \left[\frac{t}{4(t-1)}, \frac{t-1}{4(t-2)}\right]$, and that by our assumption $t \geq 3$. It is easy to check that (3.22) turns into exact equality at both end points $\mu = \frac{t}{4(t-1)}, \frac{t-1}{4(t-2)}$ of the interval (which explains our choice $\frac{3}{2}(1-2\mu)$ of the slope), hence we only have to check that the left-hand side $\theta'_{t-1}(2\mu)$ of (3.22) is concave as a function of μ. This follows from the direct computation

$$\begin{aligned}\frac{d^2\theta'_{t-1}(2\mu)}{d\mu^2} &= -\frac{3u^{1/2}(u-1)}{(4u\delta+u+1)^{3/2}\delta^{1/2}(u+1)^2}\\ &\times(128u^2\delta^2 + 64u^{3/2}\delta^{3/2}(4u\delta+u+1)^{1/2} + 16u^{3/2}\delta^{1/2}(4u\delta+u+1)^{1/2}\\ &\quad +16u^{1/2}\delta^{1/2}(4u\delta+u+1)^{1/2} + 48u^2\delta + 48u\delta + 3u^2 + 6u + 3) \leq 0,\end{aligned}$$

where $u \stackrel{\text{def}}{=} t-1$ and $\delta \stackrel{\text{def}}{=} \mu - \frac{t}{4(t-1)} \geq 0$. □

Proof of (3.21)**.** Using the previous notation (3.18), we can re-write the inequality (3.21) as

$$\frac{3(s-1)}{s(s+1)^2}(sz+\xi)(s+1-sz-\xi) \leq \frac{3}{2}(1-2\mu). \tag{3.23}$$

Since $s \leq t-2$, we have $\mu \leq \frac{s+1}{4s}$ and $z \leq 2\mu \leq \frac{s+1}{2s}$. It is easy to see that for $\mu = \frac{s+1}{4s}$ we have $\xi = \frac{s+1}{2} - sz$ and (3.23) becomes equality; also, the right-hand side is decreasing in μ. Thus, we only have to prove that its left-hand side is increasing in μ as long as $\mu \leq \frac{s+1}{4s}$. This is easy: first, ξ is clearly increasing in μ. Second, $\mu \leq \frac{s+1}{4s}$ implies $\xi \leq \frac{s+1}{2} - sz$, and then $(sz+\xi)(s+1-sz-\xi)$ is also increasing as a function of ξ. □

We have proved both bounds (3.20) and (3.21) on the derivatives, and they imply (3.19). Substituting this into (3.16), and recalling the meaning (3.12) of z, we find

$$\phi^1(K_4^1) \geq A^3\left(\frac{3}{2}(1-2\mu)\left(\frac{\phi^1(e)}{A} - \eta_{t-1}\right) + \eta_{t-1}^3 \cdot \frac{(t-2)(t-3)}{(t-1)^2}\right). \tag{3.24}$$

This conclusion holds for any $\phi^1 \in \mathrm{Hom}^+(\mathcal{A}^1, \mathbb{R})$ for which we know (3.9), (3.11). In particular, we can apply it to the random homomorphism $\boldsymbol{\phi_0^1}$; averaging the result (and recalling that $\mathbf{E}\left[\boldsymbol{\phi_0^1}(K_4^1)\right] = \phi_0(K_4)$ and $\mathbf{E}\left[\boldsymbol{\phi_0^1}(e)\right] = \phi_0(\rho) = a$ by (2.6)), we finally get the bound

$$\phi_0(K_4) \geq A^3\left(\frac{3}{2}(1-2\mu)\left(\frac{a}{A} - \eta_{t-1}\right) + \eta_{t-1}^3 \cdot \frac{(t-2)(t-3)}{(t-1)^2}\right). \tag{3.25}$$

Comparing (3.6) with (3.25) (and recalling that $h_t'(a) = \frac{3}{2}A$), we get

$$\left.\begin{aligned} & b\left(\frac{3}{2}A + 3a - 2\right) - \frac{3}{2}a(2a-1)A \\ & \geq A^3\left(\frac{3}{2}(1-2\mu)\left(\frac{a}{A} - \eta_{t-1}\right) + \eta_{t-1}^3 \cdot \frac{(t-2)(t-3)}{(t-1)^2}\right). \end{aligned}\right\} \tag{3.26}$$

For analyzing this constraint we once more recall that $\mu \in \left[\frac{t}{4(t-1)}, \frac{t-1}{4(t-2)}\right]$, and let

$$\mu = \frac{t-1}{4(t-2)} - \frac{\sigma^2}{4(t-1)(t-2)},$$

where $\sigma \in [0,1]$. Inverting the function $h_t'(a)$, we get

$$a = \frac{At(4(t-1) - A(t+1))}{4(t-1)^2}.$$

Also,

$$b = aA - \mu A^2,$$

and, solving the quadratic equation (3.17),

$$\eta_{t-1} = \frac{t-1-\sigma}{2(t-2)}.$$

Substituting all this into (3.26), we, after simplifications and cancelations, get a *polynomial* constraint of the form

$$(\sigma - 1)\frac{A^2 W(t, A, \sigma)}{8(t-1)^2(t-2)^2} \geq 0, \tag{3.27}$$

where, moreover, the polynomial $W(t, A, \sigma)$ is *linear* in A. We claim that

$$W(t, A, \sigma) > 0. \tag{3.28}$$

Indeed, due to the concaveness of h_t, $\frac{3(t-1)}{t+1} = h_t'\left(1 - \frac{1}{t+1}\right) \leq h_t'(a) \leq h_t'(1 - \frac{1}{t}) = \frac{3(t-1)}{t}$, which implies $A \in \left[\frac{2(t-1)}{t+1}, \frac{2(t-1)}{t}\right]$. Since W is linear in A, it is sufficient to check (3.28) only at the end points of this interval, and calculations show that

$$\begin{aligned} W\left(t, \frac{2(t-1)}{t+1}, \sigma\right) &= \frac{2(t-1)}{t+1}(-2\sigma^2 t + (\sigma+1)(t^2 - 3t + 4)) \\ W\left(t, \frac{2(t-1)}{t}, \sigma\right) &= 2(t-1)(-2\sigma^2 + (\sigma+1)(t-1)). \end{aligned}$$

Both quadratic (in σ) polynomials appearing here have negative leading terms, and at the end points $\sigma = 0, 1$ they attain strictly positive values $t^2 - 3t + 4,\ 2(t-2)^2,\ t-1,\ 2(t-2)$ (remember that we assumed $t \geq 3$). This proves (3.28), and, therefore, (3.27) implies $\sigma \geq 1$, and (since $\sigma \in [0,1]$), $\sigma = 1$. Which means $\mu = \frac{t}{4(t-1)}$ and $b = aA - \frac{t}{4(t-1)}A^2 = h_t(a)$.

Theorem 3.1 is proved. □

4. Latest developments

4.1. An exact bound

As observed by the referee of this paper, the standard blow-up trick provides for an easy conversion of asymptotic results into exact bounds which, in our opinion, narrows the gap between analytical and discrete methods even further. In our framework this trick can be re-casted as follows.

Every *finite* graph G gives rise to a *homomorphism* $\phi_G \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ that intuitively corresponds to its "infinite blow-up". ϕ_G can be described in two ways, one is combinatorial and another is analytical. Combinatorially, let us denote by $G^{(k)}$ the ordinary (finite) blow-up of G, that is the graph on $V(G) \times [k]$ defined by $((v,i),(w,j)) \in E(G^{(k)})$ if and only if $v \neq w$ and $(v,w) \in E(G)$. Then the sequence $\{G^{(k)}\}$ is convergent as $k \to \infty$ (that is, for every *fixed* H the sequence $p(H, G^{(k)})$ is convergent) and ϕ_G is the limit of this sequence. Analytically, every finite graph G gives rise to a zero-one valued *graphon* $W_G : [0,1] \longrightarrow \{0,1\}$ that is a *stepfunction* (see e.g. [3]). ϕ_G is the homomorphism naturally corresponding to this graphon.

Either way, the values $\phi_G(H)$ are easy to compute by an explicit formula that in the case of complete graphs has a particularly simple form

$$\phi_G(K_r) = \frac{n(n-1)\cdot\ldots\cdot(n-r+1)}{n^r} \cdot p(K_r, G) = \frac{r!}{n^r} c_r(G),$$

where $c_r(G)$ is the *number* of complete subgraphs on r vertices. Applying to the homomorphism ϕ_G our main result, we immediately get

Theorem 4.1. $f_3(m,n) \geq \frac{n^3}{6} g_3(\frac{2m}{n^2})$.

4.2. Larger values of r

After the preliminary version of this paper had been disseminated, V. Nikiforov [16] proved Conjecture 1 for $r = 4$. Like Fisher [9] and us, Nikiforov's proof very substantially uses analytical methods (although, instead of optimizing over the whole set $\mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$, he optimizes over its closed subset corresponding to the stepfunctions with a *fixed* number of steps). The problem of finding a purely combinatorial proof of Conjecture 1 remains open even in the simplest non-trivial case $r = 3,\ 1/2 \leq \rho \leq 2/3$.

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