Complexity Theory

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You may (and are mildly encouraged to) work together on solving homework problems, but please put all the names of your collaborators at the top of the assignment. Everyone must turn in his/her own independently written solution.

Shopping for solutions on the Internet is strongly discouraged. If you encounter it anyway, you must completely understand the proof, explain it in your own words and include the URL.

PDF file **prepared from a TeX source** is the preferred format. In that case you will get back your feedback in equally neat form.

Homework 1, due April 21

1. Define the *blank* (*non-blank*) complexity measure $\Phi_e(x)$ as the number of times the head of the one-tape Turing machine M_e observes the blank symbol[1] # (any other symbol, respectively) during its execution on the input x. If M_e does not halt on x, $\Phi_e(x)$ is undefined.

   Prove that both blank and non-blank complexity measures are actually abstract complexity measures, that is they satisfy Blum axioms.

2. For a language $L \subseteq \{0,1\}^*$, let

$$L^+ \stackrel{\text{def}}{=} \{\underbrace{x\#x\#\ldots\#x}_{n^{\log\log\log n}\ \text{times}} \mid x \in L,\ |x| = n\}.$$

   Clearly, $L^+ \leq_p L$.

   Prove that if $L \leq_p L^+$ then $L \in \mathsf{P}$.

[1] Recall that the computation is initialized in the state $x\#\#\#\ldots$

3. Prove that $\mathsf{SPACE}\left(n^{2023}\right)$ is *not* closed under (poly-time) Karp reductions $\leq_p$.

4. Construct an **explicit** and **direct** (that is, bypassing the Cook-Levin theorem) poly-time Karp reduction from VERTEX COVER to SATISFIABILITY.

5. Prove that the following two modifications of the SUBSET VECTOR SUM PROBLEM are NP-complete.

   (a) `INSTANCE:` a list[2] of integer vectors $v_1, \ldots, v_n \in \mathbb{Z}^m$, a target vector $v \in \mathbb{Z}^m$ and an integer k.

   `Question:` does there exist $S \subseteq [n]$ with $|S| = k$ such that $\sum_{i \in S} v_i = v$?

   (b) `INSTANCE:` a list of integer vectors $v_1, \ldots, v_n$.

   `Question:` does there exist a **non-empty** $S \subseteq [n]$ such that $\sum_{i \in S} v_i = 0$?

[2] in all these problems, vectors $v_1, \ldots, v_n$ need **not** necessarily be distinct