Complexity Theory

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You may (and are mildly encouraged to) work together on solving homework problems, but please put all the names of your collaborators at the top of the assignment. Everyone must turn in his/her own independently written solution.

Shopping for solutions on the Internet is strongly discouraged. If you encounter it anyway, you must completely understand the proof, explain it in your own words and include the URL.

PDF file **prepared from a TeX source** is the preferred format. In that case you will get back your feedback in equally neat form.

Homework 3, due May 19

1. Prove that $L_{\mathrm{mon}}(f_n) \leq 2^n$ for any monotone Boolean function in n variables. The size of a monotone formula is computed as the number of $\wedge, \vee$-gates; constants 0,1 and variables are allowed for free.

2. Let $f_1, \ldots, f_m$ be non-constant functions with *pairwise disjoint sets of variables*. Prove that:

   (a) $L_{\{\neg,\wedge,\vee\}}(f_1 \oplus \ldots \oplus f_m) \geq \frac{1}{2} \sum_{i=1}^m L_{\{\neg,\wedge,\vee\}}(f_i)$;
   (b) $L_{\{\neg,\wedge,\vee\}}(f_1 \oplus \ldots \oplus f_m) \geq \Omega(m^2)$.

3. Let now $f_1, \ldots, f_m$ be arbitrary functions. Prove that

$$BP(f_1 \oplus \ldots \oplus f_m) \leq 2 \sum_{i=1}^m BP(f_i) - \max_{i \in [m]} BP(f_i).$$

4. Prove that MOD_4 is in $\mathsf{AC}^0[2]$, i.e. can be computed by constant depth polynomial size circuits over $\{\neg, \wedge, \vee, \oplus\}$.

5. Consider the "Toeplitz 2023-diagonal" $N \times N$ matrix A, where

$$A(x, y) = \begin{cases} 1 & \text{if } |x - y| \leq 2023; \\ 0 & \text{if } |x - y| > 2023. \end{cases}$$

Compute its deterministic and randomized (private randomness) communication complexities within a constant multiplicative factor.