Complexity Theory

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You may (and are mildly encouraged to) work together on solving homework problems, but please put all the names of your collaborators at the top of the assignment. Everyone must turn in his/her own independently written solution.

You (obviously) have to prove all your answers, and everything that was stated in class can be used without a proof unless explicitly forbidden in the statement.

Shopping for solutions on the Internet is strongly discouraged. If you encounter it anyway, you must completely understand the proof, explain it in your own words and include the URL.

PDF file **prepared from a TeX source** is the preferred format. In that case you will get back your feedback in equally neat form.

Homework 1, due April 25

1. Formulate the Space Gap Theorem and explain why it follows from the Time Gap Theorem.

2. Prove that the class $\mathsf{NSPACE}(n^{2025})$ is not closed under many-one Karp reductions. **Note.** You are **not** allowed to use *Non-deterministic* Space Hierarchy Theorem without giving a complete proof even if you manage to find it somewhere.

3. Let NP_{2025} be the class of all languages L for which there exists a non-deterministic poly-time Turing machine M such that:

   (a) if $x \notin L$ then the computation of M on x does not have any accepting paths.

(b) if $x \in L$ then the computation of M on x has at least 2025 accepting paths.

Prove that $\mathsf{NP}_{2025} = \mathsf{NP}$.

4. Given a CNF $\phi(x_1, \ldots, x_n) = \{C_1, \ldots, C_m\}$, its *incidence graph* is the bipartife graph G_ϕ on $[n] \times [m]$ such that $(i, j) \in E(G_\phi)$ iff x_i occurs in C_j (negatively or positively). Let TREE-SAT be the restriction of SATISFIABILITY to those ϕ for which G_ϕ is a tree.

   Prove that TREE-SAT is in P.

5. A *perfect matching* in a graph with $2n$ vertices is a set of n edges that do not have any vertices in common. PERFECT MATCHING is the language that consists of all graphs with even number of vertices that possess at least one perfect matching.

   Construct a many-one Karp reduction from PERFECT MATCHING to CLIQUE.

   Note. The reduction is supposed to be explicit and combinatorial. You are not allowed to use without giving a complete proof either Cook-Levin theorem or the fact that PERFECT MATCHING is actually in P.