Complexity Theory

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You may (and are mildly encouraged to) work together on solving homework problems, but please put all the names of your collaborators at the top of the assignment. Everyone must turn in his/her own independently written solution.

You (obviously) have to prove all your answers, and everything that was stated in class can be used without a proof unless explicitly forbidden in the statement.

Shopping for solutions on the Internet is strongly discouraged. If you encounter it anyway, you must completely understand the proof, explain it in your own words and include the URL.

PDF file **prepared from a TeX source** is the preferred format. In that case you will get back your feedback in equally neat form.

Homework 3, due May 23

1. Let us call a language $L \subseteq \{0,1\}^*$ *robust* if every string in L has at most 2025 alternations of 0 and 1. Let ROBUST be the class of all robust languages.

   Prove that $\mathsf{P}^{\mathsf{ROBUST}} = \mathsf{P}/poly$.

2. Prove the analogue of Spira's theorem for the basis $\{MAJ_{2025}\}$ (Boolean constants 0 and 1 are also allowed).

3. Prove that

$$L_{\{\neg,\wedge,\vee\}}(MAJ_n(x_{11},\ldots,x_{1n}) \oplus MAJ_n(x_{21},\ldots,x_{2n}) \oplus \ldots \oplus MAJ_n(x_{n1},\ldots,x_{nn})) \geq \Omega(n^4).$$

4. Consider branching programs with the following restrictions. Nodes are labelled as $(v_{ij} \mid i \in [S],\ j \in [2025])$ and every edge should have the form $\langle v_{ij}, v_{i+1,j'} \rangle$ with $j' \geq j$.

   Prove that every b.p. of this form computing $x_1 \oplus \ldots \oplus x_n$ must have size $\exp(\Omega(n))$.

5. Recall that the (Boolean) permanent function $\mathrm{PERM}_n\,(x_{ij} \mid 1 \leq i, j \leq n)$ and the clique function $\mathrm{CLIQUE}_{k,n}\,(y_{ij} \mid 1 \leq i < j \leq n)$ are given as

$$\mathrm{PERM}_n \stackrel{\mathrm{def}}{=} \bigvee_{\sigma \in S_n} \bigwedge_{i=1}^{n} x_{i,\sigma(i)}$$

$$\mathrm{CLIQUE}_{k,n} \stackrel{\mathrm{def}}{=} \bigvee_{K \in \binom{[n]}{k}} \bigwedge_{\substack{i,j \in K \\ i<j}} y_{ij}.$$

   Prove that $C_{\mathrm{mon}}(\mathrm{PERM}_n) \leq 2C_{\mathrm{mon}}(\mathrm{CLIQUE}_{n,n^2})$, where C_{mon} is the monotone circuit size.