Complexity Theory B

Instructor: Alexander Razborov, University of Chicago
razborov@math.uchicago.edu
Course Homepage:
www.cs.uchicago.edu/~razborov/teaching/ComplexityB/spring17.html

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You are encouraged to work together on solving homework problems, but please put all the names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. Shopping for solutions on-line is strongly discouraged. If you nonetheless decide to venture this, as the very least you must completely understand the proof, explain it in your own words and include the URL.

The deadline below pertains to e-mail submissions as a PDF file prepared from a TeX source. The grace period lasts until 10am on the day following the deadline, later submissions shall be penalized at the rate 10% per hour.

Homework 1, due May 3

1. Let us call a language $L \subseteq \{0,1\}^*$ *diadic* if for every $x \in L$ and every $i \leq |x|$ with $x_i = 1$, i is a power of two.

   Prove that the class of languages that can be recognized by a polynomial time oracle machine with a diadic oracle is equal to $\mathsf{P}/poly$.

2. Prove that the problem of testing if a given undirected graph is bipartite belongs to the class co-SL (and hence, by Reingold's result, to $\mathsf{LOGSPACE}$).

3. A *formal complexity measure* is a functional $\mu : F_n \longrightarrow \mathbb{R}_{\geq 0}$ (F_n is the set of all Boolean functions $f : \{0,1\}^n \longrightarrow \{0,1\}$ in n variables) such that:

- $\mu(0), \mu(1), \mu(x_i), \mu(\neg x_i) \leq 1$ ($i \in [n]$);
- $\mu(f \wedge g) \leq \mu(f) + \mu(g)$ and $\mu(f \vee g) \leq \mu(f) + \mu(g)$ ($f, g \in F_n$).

(a) Design a formal complexity measure μ such that for some $f \in F_n$ and some restriction ρ, $\mu(f|_\rho)$ is exponentially larger than $\mu(f)$.

(b) What additional *natural* condition one should impose on formal complexity measures so that this does not happen and $\mu(f|_\rho) \leq O(\mu(f))$ always holds?

4. Let $k > 2$ and suppose $\mathcal{C} = (C_1, \ldots, C_m)$ is a Boolean k-CSP (that is, all constraints involve k $\{0,1\}$-valued variables) over n variables. Show an efficient (defined as polynomial in $m, n, 2^k$) algorithm that transform $\mathcal{C}$ to a 2-CSP $\mathcal{C}'$ (not necessarily Boolean) such that:

   (a) If $\mathcal{C}$ is satisfiable then so is $\mathcal{C}'$;

   (b) $\mathrm{UNSAT}(\mathcal{C}') \geq \Omega(\mathrm{UNSAT}(\mathcal{C})/k)$ (recall that $\mathrm{UNSAT}(\mathcal{C})$ is the minimum fraction of unsatisfied constraints under any assignment to the variables).

5. Prove that $\mathrm{MOD}_{2017}(x_1, \ldots, x_n)$ is computable by a bounded depth polynomial size circuit in the basis $\{0, 1, \neg, \mathrm{MAJ}\}$.