Complexity Theory B

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You are encouraged to work together on solving homework problems, but please put all the names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. Shopping for solutions on-line is strongly discouraged. If you nonetheless decide to venture this, as the very least you must completely understand the proof, explain it in your own words and include the URL.

The deadline below pertains to e-mail submissions as a PDF file prepared from a TeX source. The grace period lasts until 10am on the day following the deadline, later submissions shall be penalized at the rate 10% per hour.

Homework 2, due May 17

1. Determine the order of magnitude of $\mathrm{Corr}(x_1 \oplus \ldots \oplus x_n, \mathrm{MAJ}_n)$.
2. Recall that for a function $f : X \times Y \longrightarrow \{0, 1\}$ and $a \in \{0, 1\}$, $\chi_a(f)$ is the minimal number of *disjoint* rectangles covering $f^{-1}(a)$.

   Prove that $\chi_0(f) \leq \exp(O(\log^2 \chi_1(f)))$.
3. Consider the "Toeplitz 2017-diagonal" $N \times N$ matrix A, where

$$A(x, y) = \begin{cases} 1 & \text{if } |x - y| \leq 2017; \\ 0 & \text{if } |x - y| > 2017. \end{cases} \tag{1}$$

   Compute its deterministic and randomized[1] communication complexities within a constant multiplicative factor.

[1] Private randomness – transmitting random bits is not free

4. Let $A(x_1, \ldots, x_k) : \{0,1\}^k \longrightarrow \{0,1\}$ outputs 1 if and only if $x_1 = x_2 = \ldots = x_k$. Consider the function $f : (\{0,1\}^n)^k \longrightarrow \{0,1\}$ given by $f(x_{ij}) = \oplus_{i=1}^n A(x_{i1}, \ldots, x_{ik})$.

   Prove that $C_k(f) \leq O(k)$ (C_k is the communication complexity in the number-on-forehead model).

5. Let us call a Boolean string $a_1, \ldots, a_n$ *mid-Western* if there exists at least one index c such that $a_c = a_{c+1} = 1$ while $a_i = 0$ for any other $i \in [n]$ with $|c - i| \leq \sqrt{n}$. Let $MW_n(x_1, \ldots, x_n)$ be the characteristic function of the set of all mid-Western strings.

   Determine its sensitivity $\mathsf{s}(MW_n)$ and block-sensitivity $\mathsf{bs}(MW_n)$ within a multiplicative constant.