Complexity Theory B

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You are encouraged to work together on solving homework problems, but please put all the names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. If you find a solution on-line, you must completely understand the proof, explain it in your own words and include the URL.

The deadline below pertains to e-mail submissions as a PDF file prepared from a TeX source. The grace period lasts until 10am on the day following the deadline, later submissions shall be penalized at the rate 10% per hour.

Homework 3, due May 31

1. Prove that $L(f) \geq \Omega(\mathsf{as}(f)^2)$, where $L(f)$ is the formula size in the de Morgan basis and $\mathsf{as}(f)$ is the average sensitivity of f.

2. Recall that $\widetilde{\deg}_\epsilon(f)$ is the approximate degree of a Boolean function f defined as the minimal possible degree of a polynomial p such that $\max_{x\in\{-1,1\}^n} |p(x) - f(x)| \leq \epsilon$.

   (a) Prove that $\widetilde{\deg}_{0.01}(f) \leq O(\widetilde{\deg}_{1/3}(f))$.

   (b) Will the analogous statement be true for ℓ_2-norm instead of ℓ_∞-norm?

3. Let $\mu : \{-1,1\}^n \longrightarrow \mathbb{R}$ be given by

$$\mu(x) \stackrel{\text{def}}{=} p^{\frac{n+\delta(x)}{2}}(1-p)^{\frac{n-\delta(x)}{2}},$$

where $\delta(x) = x_1 + \ldots + x_n$ and $p \in [0,1]$. Compute its Fourier expansion $\widehat{\mu}$ as a **closed-form expression** (no factorials, binomial coefficients, unexpanded sums or products etc.).

4. Let $L_k(f_1, \ldots, f_m)$ be the multiplicative (only multiplications count) algebraic complexity over the field k.

   (a) For $f_1, \ldots, f_m \in \mathbb{R}[x_1, \ldots, x_n]$, prove that $L_{\mathbb{R}}(f_1, \ldots, f_m) \leq 3L_{\mathbb{C}}(f_1, \ldots, f_m)$.

   (b) For any n, give an example of a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ such that $L_{\mathbb{C}}(f) = \lceil n/2 \rceil$ but $L_{\mathbb{R}}(f)$ "seems to be"[1] n.

5. Prove that the clique polynomials $\mathrm{Cl}_{k,n}$ in the variables $(\, x_{ij} \mid 1 \leq i < j \leq n\,)$ defined by

$$\mathrm{Cl}_{k,n}(\vec{x}) \stackrel{\text{def}}{=} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \ \prod_{1 \leq \nu < \mu \leq k} x_{i_\nu i_\mu}$$

   are in VNP.

[1] I did not mention in class the method that actually allows to prove that.