Honors Discrete Mathematics

Instructor: Alexander Razborov, University of Chicago
razborov@math.uchicago.edu
Course home page: www.cs.uchicago.edu/~razborov/teaching/autumn16.html

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Prove all of your answers. If you work with others put their names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. Homework is due at the beginning of class *unless* submitted by e-mail as a PDF file prepared from a TeX source. Electronic submissions conforming to these standards (no scans please) are accepted until Wednesday midnight at lenacore@uchicago.edu.

Homework 3, due October 26

1. For two ideals I and J in a commutative ring, let

$$I + J \stackrel{\text{def}}{=} \{a + b \mid a \in I,\ b \in J\}$$
$$I \cdot J \stackrel{\text{def}}{=} \{a_1b_1 + a_2b_2 + \cdots + a_nb_n \mid n \in \mathbb{N},\ a_1, \ldots, a_n \in I,\ b_1, \ldots, b_n \in J\}.$$

   Prove that $I+J$ and $I\cdot J$ are also ideals and that $I\cdot(J+K) = I\cdot J+I\cdot K$ for any three ideals I, J, K.

2. Prove that $\gcd(3, 1 + \sqrt{5}) = 1$ in $\mathbb{Z}[\sqrt{5}]$.

3. Construct two equivalence relations $\approx_1$ and $\approx_2$ on the set $[2016] \stackrel{\text{def}}{=} \{1, 2, \ldots, 2016\}$ such that:

   (a) Every equivalence class of either $\approx_1$ or $\approx_2$ has at most two elements;

   (b) Every equivalence relation $\approx$ that is *coarser*[1] than both $\approx_1$ and $\approx_2$ is trivial, i.e. consists of all possible 2016^2 pairs.

[1] This means that any equivalence class of either $\approx_1$ or $\approx_2$ is contained in an equivalence class of $\approx$.

4. How many ideals[2] does the ring $\mathbb{Z}_{2048}$ possess?

5. Solve the following system of congruences:

$$\begin{cases} x \equiv 1 \pmod{2} \\ x \equiv 1 \pmod{3} \\ x \equiv 1 \pmod{5} \\ x \equiv 2 \pmod{7} \\ x \equiv 2 \pmod{11}. \end{cases}$$

[2]including the two trivial ideals, 0 and $\mathbb{Z}_{2048}$ itself