Honors Discrete Mathematics

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`Prove all of your answers. If you work with others put their names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. Homework is due at the beginning of class` *`unless`* `submitted by e-mail as a PDF file prepared from a TeX source. Electronic submissions conforming to these standards (no scans please) are accepted until Wednesday midnight at lenacore@uchicago.edu.`

Homework 4, due November 2

1. Give a closed-form expression for the number of ordered pairs $\langle A, B\rangle$, where $A, B \subseteq [n]$ are such that $|A \setminus B| = 17$.

2. Let us call a function $f : X \longrightarrow Y$ *gorgeous* if for any function $g : Z \longrightarrow Y$ with $|Z| \leq 17$ there exists an injective function $h : Z \rightarrowtail X$ such that $g = f \circ h$. Let us call f *co-gorgeous*[1] if for any $g : X \longrightarrow Z$, again with $|Z| \leq 17$, there exists a surjective $h : Y \twoheadrightarrow Z$ such that $g = h \circ f$.

   (a) Describe the set of those integers n for which there exists a gorgeous function $f : [100] \longrightarrow [n]$.

   (b) Describe the set of those integers n for which there exists a co-gorgeous function $f : [100] \longrightarrow [n]$.

3. What is the free term (i.e., the coefficient in front of 1) in the Laurent polynomial $(x^3 + ix^{-2})^{100}$? ($i = \sqrt{-1}$)

[1] Well, what else would you expect?

4. Prove that
$$\sum_{k=0}^{n}\sum_{\ell=0}^{n}\binom{n}{k}\binom{n}{\ell}\binom{n}{k+\ell}=\binom{3n}{n}.$$

5. Let A_i $(i \in \mathbb{Z}_6)$ be sets such that for any $i \in \mathbb{Z}_6$ we have $|A_i| = 100$, $A_i \cap A_{i+1} = \emptyset$, $|A_i \cap A_{i+2}| = 10$, $|A_i \cap A_{i+3}| = 20$ and $|A_i \cap A_{i+2} \cap A_{i-2}| = 1$. Compute $|A_0 \cup \ldots \cup A_5|$.