Honors Discrete Mathematics

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Prove all of your answers. Factorials $n!$ and, respectively, binomial coefficients $\binom{n}{k}$ for $n \geq 7$ may be left unexpanded. If you work with others put their names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. Homework is due at the beginning of class *unless* submitted by e-mail as a PDF file prepared from a TeX source. Electronic submissions conforming to these standards (no scans please) are accepted until Wednesday midnight at lenacore@uchicago.edu.

Homework 7, due November 23

1. How many integer (*not* necessarily positive) solutions are there to the equation
$$x_1 \cdot x_2 \cdot x_3 \cdot x_4 = 2^{20} \cdot 3^{16}?$$
Order matters.

2. Let $X_1, \ldots, X_6$ be independent unbiased Bernoulli random variables. Compute the conditional expectation $E(X_1+X_2+\ldots+X_6|(X_1+X_2+X_3 \geq 1) \vee (X_4+X_5+X_6 \geq 1))$.

3. For a permutation $\sigma : [n] \longrightarrow [n]$ define its *median number* $\mathrm{md}(\sigma)$ as the number of those triples (i, j, k) $(1 \leq i < j < k \leq n)$ for which $\sigma(j)$ lies between $\sigma(i)$ and $\sigma(k)$ (that is, either $\sigma(i) < \sigma(j) < \sigma(k)$ or $\sigma(k) < \sigma(j) < \sigma(i)$). Prove that $p\left(\mathrm{md}(\sigma) \geq \frac{n^3}{10}\right) \leq \frac{5}{9}$, where σ is chosen uniformly at random from the set of all permutations of $[n]$.

4. Assume that $d_1, \ldots, d_n$ is the degree sequence of a simple graph. Prove that the following sequences are also degree sequences of a simple graph (order does not matter):

   (a) $d_1 + 1, d_1 + 1, d_2, d_2, d_3, d_3, \ldots, d_n, d_n$;

   (b) $k, d_1 + 1, d_2 + 1, \ldots, d_k + 1, d_{k+1}, \ldots, d_n$;

   (c) $n - 1 - d_1, n - 1 - d_2, \ldots, n - 1 - d_n$.

5. Consider the graph in which vertices are squares of the chess (8×8) board, and edges are made by adjacent (that is, sharing one side) pairs of squares. How many different paths of length (= the number of edges) 14 are there from the lower left corner to the upper right one?