Honors Discrete Mathematics

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Prove all of your answers. If you work with others put their names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged: if you do it nonetheless, you *must* cite your source and, as the very least, explain the solution in your own words.

Homework is due at the beginning of class *unless* submitted by e-mail as a PDF file prepared from a TeX source. Electronic submissions conforming to these standards (no scans please!) are encouraged and accepted until Wednesday midnight at the Canvas.

Homework 2, due October 18

1. (**challenge problem**)

   (a) Is it true that for every non-empty countable linearly ordered set $(S, \leq)$, the lexicographic product $(S, \leq) \times (\mathbb{Q} \ \leq)$ is isomorphic to $(\mathbb{Q}, \leq)$?

   (b) Is it true that for every non-empty countable linearly ordered set $(S, \leq)$, the lexicographic product $(\mathbb{Q} \ \leq) \times (S, \leq)$ is isomorphic to $(\mathbb{Q}, \leq)$?

   Note. The stern Internet warning in the header pertains only to *solutions*. Occasionally, for challenge problems you will need to learn additional *facts* beyond our curriculum, and this is of course encouraged. You will have to provide a reference in any case, though.

2. Prove that positive integers a and b are relatively prime (that is, $\gcd(a, b) = 1$) if and only if the following statement holds. For any positive integer x, $a|x$ and $b|x$ imply $ab|x$.

3. Find polynomials $x(n), y(n)$ such that

$$(n^2 - n - 1)x(n) + (n^2 + 3n + 3)y(n) = n^2 - n + 1.$$

4. Let X be a set and Λ be the collection of all its subsets. Let us try to define the structure of a commutative ring on Λ by letting $0 \stackrel{\text{def}}{=} \emptyset,\ 1 \stackrel{\text{def}}{=} X,\ A + B \stackrel{\text{def}}{=} A \cup B,\ A \cdot B \stackrel{\text{def}}{=} A \cap B.$

   (a) Which axiom of a commutative ring will fail in this attempt?

   (b) Write down at least one reasonable axiom that will hold for this structure and will not hold for arbitrary commutative rings.

5. Find polynomials $p(n), q(n), r(n)$ of degree 2017 each such that

$$p(n) + q(n)r(n) = (p(n) + q(n))(p(n) + r(n)).$$