Honors Discrete Mathematics

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`Prove all of your answers. If you work with others put their names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged: if you do it nonetheless, you *must* cite your source and, as the very least, explain the solution in your own words.`

`Homework is due at the beginning of class` *`unless`* `submitted by e-mail as a PDF file prepared from a TeX source. Electronic submissions conforming to these standards (no scans please!) are encouraged and accepted until Wednesday midnight at the Canvas.`

Homework 5, due November 8

1. **(challenge problem)** For a finite set of positive integers A, denote by $f(A)$ the product of all primes that divide at least one element of A. Fix an integer n, and for $B \subseteq [n]$ let
$$f^*(B) \stackrel{\text{def}}{=} \sum_{[n]\supseteq A\supseteq B} f(A).$$
Prove that for any $A \subseteq [n]$ we have
$$\sum_{A\subseteq B\subseteq [n]} (-1)^{|B\setminus A|} f^*(B) = f(A).$$

2. How many ways are there to choose six cards out of a standard deck of 52 cards in such a way that all four suits are represented in the selection? `Note. You may use calculators in # 2 and #3 to evaluate`

`close form expressions provided the expressions themselves are clearly written down.`

3. The pool game consists of seven indistinguishable[1] striped balls, seven indistinguishable solid balls, one white ball and one black ball. A *position* is a placement of these balls into six distinguishable pockets such that the white ball and the black one are in different holes. How many different positions are there in this game?

4. Prove that $p_n(m)$ (the number of partitions of m using *at most* n numbers) is equal to the number of partitions of $m+n$ using *exactly* n numbers.

5. Five points are chosen inside a unit square.

   (a) Prove that there is a pair of them at the distance $\leq \frac{\sqrt{2}}{2}$ of each other.

   (b) Is this bound tight? In other words, can $\frac{\sqrt{2}}{2}$ be replaced by any smaller number?

[1]for the purpose of this exercise we disregard the numbers actually written on the balls