Honors Discrete Mathematics

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   Prove all of your answers.  If you work with others put their
names clearly at the top of the assignment.  Everyone must turn
in their own independently written solutions.  Shopping for solutions
on the Internet is strongly discouraged:  if you do it nonetheless,
you *must* cite your source and, as the very least, explain the
solution in your own words.
   Homework is due at the beginning of class unless submitted by
e-mail as a PDF file prepared from a TeX source.  Electronic submissions
conforming to these standards (no scans please!)  are encouraged
and accepted until Wednesday midnight at the Canvas.
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Homework 7, due November 22

1. Compute the expectation of X^3/Y, where X and Y are the outcomes of two fair independent (cubic) dice.

2. Let X be a *non-negative* random variable. Prove that

$$E(X) \geq \sum_{k=0}^{\infty} 2^{k-1} p(X \geq 2^k).$$

   `Note. If you feel uncomfortable with infinite series, you may assume here that` X `takes on only finitely many values with non-zero probability.`

3. The *mean deviation* $MD(X)$ of a random variable X is defined as $E(|X-c|)$, where $c = E(X)$ is the expectation of X.

(a) Prove that for any two random variables X and Y on the same sample space, $MD(X+Y) \leq MD(X) + MD(Y)$.

(b) Prove that if X and Y are additionally known to be independent, then this inequality is *always* strict, unless one of the variables X, Y is trivial (that is, takes one fixed value with probability 1).

4. **(challenge problem)** Recall that random variables X and Y on the same sample space are *positively correlated* if their covariance $\mathrm{cov}(X, Y) \stackrel{\mathrm{def}}{=} E(XY) - E(X)E(Y)$ is strictly positive.

   Prove or disprove the following: if X, Y are positively correlated and Y, Z are positively correlated then X, Z are positively correlated.

5. Let $X \stackrel{\mathrm{def}}{=} |\mathrm{im}(f)|$, where $f : [m] \longrightarrow [n]$ is picked uniformly at random among all functions from $[m]$ to $[n]$, and let $c \stackrel{\mathrm{def}}{=} E(X)$. Prove that

$$p(X \geq 2c) \leq 1/c - 1/n.$$