Honors Discrete Mathematics

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Course Homepage: www.cs.uchicago.edu/~razborov/teaching/autumn18.html

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   Prove all of your answers with reasonable degree of mathematical
rigor.  If you work with others put their names clearly at the top
of the assignment.  Everyone must turn in their own independently
written solutions.  Shopping for solutions on the Internet is strongly
discouraged:  if you do it nonetheless, you *must* cite your source
and, as the very least, explain the solution in your own words.
   Homework is due at the beginning of Wednesday class unless submitted
by e-mail as a PDF file prepared from a TeX source.  Electronic
submissions conforming to these standards (no scans please!)  are
encouraged and accepted until Wednesday midnight by Leo at lenacore@uchicago.edu.
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Homework 2, due October 24

1. Prove that for a positive integer n, $\gcd(n^2+n+1, n^2-1)$ may take on only two values: either 1 or 3.

2. Similarly to the case $k = 2$, define the *greatest common divisor* $\gcd(a_1, \ldots, a_k)$ of the integers $a_1, \ldots, a_k$ as the largest integer d such that $d|a_1, \ldots, d|a_k$.

   Give an example of integers a, b, c, d such that $\gcd(u, v, w) = 1$ for any three-element subset $\{u, v, w\} \subseteq \{a, b, c, d\}$, while $\gcd(u, v) > 1$ for any 2-element subset.

3. How many different ideals (including trivial ones) does the ring $\mathbb{Z}_{1000}$ have?

4. Let R be a commutative ring, and $\mathcal{I}$ be the set of all its ideals (including trivial ones). For $I, J \in \mathcal{I}$, let

$$\begin{aligned} I + J &\stackrel{\text{def}}{=} \{a + b \mid a \in I,\ b \in J\} \\ I \cdot J &\stackrel{\text{def}}{=} \{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \mid n \in \mathbb{N},\ a_1, \ldots, a_n \in I,\ b_1, \ldots, b_n \in J\}. \end{aligned}$$

Prove that $I + J$ and $I \cdot J$ are also ideals and that these two operations on $\mathcal{I}$ satisfy[1] all axioms of a commutative ring except for one.

5. Give an example of an ideal in $\mathbb{R}[x, y]$ that can be generated by three elements but can not be generated by two elements.

[1] you will have to figure out first what ideals play roles of 0 and 1