Honors Discrete Mathematics

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Course Homepage: www.cs.uchicago.edu/~razborov/teaching/autumn18.html

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   Prove all of your answers with reasonable degree of mathematical
rigor.  If you work with others put their names clearly at the top
of the assignment.  Everyone must turn in their own independently
written solutions.  Shopping for solutions on the Internet is strongly
discouraged:  if you do it nonetheless, you *must* cite your source
and, as the very least, explain the solution in your own words.
   Homework is due at the beginning of Wednesday class unless submitted
by e-mail as a PDF file prepared from a TeX source.  Electronic
submissions conforming to these standards (no scans please!)  are
encouraged and accepted until Wednesday midnight by Leo at lenacore@uchicago.edu.
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Homework 4, due November 7

1. Solve the following system of congruences:

$$\begin{cases} x \equiv 1 \pmod{2} \\ x \equiv 2 \pmod{3} \\ x \equiv 1 \pmod{5} \\ x \equiv 2 \pmod{7} \\ x \equiv 1 \pmod{11} \\ x \equiv 2 \pmod{13}. \end{cases}$$

2. Let X, Y be non-empty finite sets. Call a function $f : X \longrightarrow Y$ *beautiful* if for any function $g : Z \longrightarrow Y$ with $|Z| \leq 2018$ there exists an injective function $h : Z \rightarrowtail X$ such that $g = f \circ h$. Call f *co-beautiful*

if for any $g : X \longrightarrow Z$ with $|Z| \leq 2018$, there exists a surjective $h : Y \twoheadrightarrow Z$ such that $g = h \circ f$.

Give an *intrinsic*[1] description of beautiful and co-beautiful functions.

3. Give a close form expression for the number of *unordered* (that is, (A, B) and (B, A) are considered the same) pairs (A, B) such that $A \cup B = [n]$.

4. Prove the identity

$$\sum_{k=2018}^{n} \binom{k}{2018}\binom{n}{k} = \binom{n}{2018} \cdot 2^{n-2018} \ (n \geq 2018).$$

5. Let A_i ($i \in \mathbb{Z}_6$) be sets such that for any $i \in \mathbb{Z}_6$ we have $|A_i| = 50$, $|A_i \cap A_{i+1}| = 20$, $A_i \cap A_{i+2} = \emptyset$ and $|A_i \cap A_{i+3}| = 5$, where all additions are $\mod 6$. Compute $|A_0 \cup \ldots \cup A_5|$.

[1] contact us if unsure about the meaning of this word