Honors Discrete Mathematics

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Autumn Quarter, 2018

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   Prove all of your answers with reasonable degree of mathematical
rigor.  If you work with others put their names clearly at the top
of the assignment.  Everyone must turn in their own independently
written solutions.  Shopping for solutions on the Internet is strongly
discouraged:  if you do it nonetheless, you *must* cite your source
and, as the very least, explain the solution in your own words.
   Homework is due at the beginning of Wednesday class unless submitted
by e-mail as a PDF file prepared from a TeX source.  Electronic
submissions conforming to these standards (no scans please!)  are
encouraged and accepted until Wednesday midnight by Leo at lenacore@uchicago.edu.
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Homework 8, due December 5

1. Consider the random variable $i(\sigma)$, where $\sigma \in S_n$ is a random permutation of $[n]$ and $i(\sigma)$ is its *inversion number* defined as the number of pairs $1 \leq i < j \leq n$ for which $\sigma(i) > \sigma(j)$.

   (a) Compute the variance of this random variable as a closed-form expression.

   (b) Prove that
   $$p(i(\sigma) \geq n^2/3) \leq C \cdot n^{-1},$$
   where $C > 0$ is an absolute constant.

2. Define a random graph $G(n, p)$ on the set of n vertices as follows: for each pair of vertices u, v, $\{u, v\}$ is declared to be an edge with probability p independently of all others[1]. What is the expected number of

[1] This model is often called *Erdős–Rényi model.*

cycles C_4 in this graph? For the purpose of this exercise, two cycles that are obtained from each other by rotations and reflections are considered the same and the cycles need not necessarily be *induced* (i.e., may contain diagonals).

3. Let $a_1, \ldots, a_r \in \mathbb{Z}_n$ be pairwise distinct. Define the undirected (but not a priori simple) graph[2] $G = (\mathbb{Z}_n, E)$, where

$$E \stackrel{\text{def}}{=} \{e_{xi} \mid x \in \mathbb{Z}_n,\ 1 \le i \le r\}$$

and the edge e_{xi} has endpoints x and $x + a_i$.

When is this graph simple? When is it connected?

4. Assume that $d_1, \ldots, d_n$ is the degree sequence of a simple graph. Prove that the following sequences are also degree sequences of a simple graph (order does not matter):

   (a) $n-1-d_1, n-1-d_2, \ldots, n-1-d_n$;

   (b) $d_1+1, d_1+1, d_2+1, d_2+1, \ldots, d_{k-1}+1, d_{k-1}+1, d_k, d_k, \ldots, d_n, d_n$;

   (c) $2d_1, d_2, d_2, \ldots, d_n, d_n$.

5. Prove or disprove the following. If two simple graphs G and H have the same degree sequences then they also contain the same number of simple paths P_3 (that is, paths of length 2).

[2]Graphs of this form are called *Cayley graphs.*